This chapter fully exhibits the relationship between the continuous wavelet transforms discussed in Chapter 2 with the Plancherel formula. A first instance of the connection was discernible in Remark 3.34 dealing with discrete series representations of unimodular groups: Computing the constant $c_\pi$ governing the admissibility condition for such representations $\pi$ was found to be equivalent to computing the Plancherel measure of the set $\{\pi\}$, and the wavelet transform was found to be a particular case of inverse Plancherel transform. These observations are systematically expanded in the course of this chapter. We thus start out by discussing Fourier inversion, first as a mapping between a direct integral space $B_1^\oplus$ of trace class operators and the Fourier Algebra $A(G)$. We then prove a Plancherel inversion formula (Theorem 4.15), from which the $L^2$-convolution Theorem 4.18 follows immediately. The same argument as for the toy example then yields admissibility conditions from the convolution theorem (Theorem 4.20). We characterize when admissible vectors exist (Theorem 4.22). Remark 4.30 offers a strategy for the solution of $T4$, sketching a systematic approach to treat arbitrary representations via direct integral theory. In the unimodular case it can actually be shown that the scheme from 4.30 characterizes the canonical Plancherel measure. We also discuss briefly how the scheme relates to the type I assumption. The final section of the chapter is devoted to a short diversion treating Wigner functions associated to nilpotent Lie groups.

The standing assumptions throughout this chapter are: $G$ and $N = \text{Ker}(\Delta_G)$ are type I, with $N$ regularly embedded.

4.1 Fourier Inversion and the Fourier Algebra*

The natural domain for Fourier and Plancherel inversion formulae is given by the Fourier algebra $A(G)$ and its counterpart $B_1^\oplus$ on the Fourier side. In order to motivate the latter space, recall the situation over the reals: Formally, the inverse Plancherel transform of $f \in L^2(\mathbb{R})$ is given as
\[ f(x) = \int_{\mathbb{R}} \hat{f}(\omega)e^{2\pi i x \omega} d\omega, \]

and it is a well-known fact in Fourier analysis that this equation holds rigourously pointwise almost everywhere whenever \( \hat{f} \in L^1(\mathbb{R}) \), which is the natural condition to ensure absolute convergence of the integral. The analogous formula for general locally compact groups will be

\[ a(x) = \int_{\hat{G}} \text{trace}(A_{\sigma}(x)^*) d\nu_G(\sigma). \quad (4.1) \]

The operator fields \( A \) for which this formula makes sense constitute the Banach space \( B_1^\oplus \) defined in the next lemma. Its proof is standard and therefore omitted.

**Lemma 4.1.** Let \( B_1^\oplus \) be the space of measurable fields \( (B(\sigma))_{\sigma \in \hat{G}} \) of trace class operators, for which the norm

\[ \|B\|_{B_1^\oplus} := \int_{\hat{G}} \|B(\sigma)\|_1 d\nu_G(\sigma) \]

is finite. Here we identify operator fields which agree \( \nu_G \)-almost everywhere. Then \( (B_1^\oplus, \|\cdot\|_{B_1^\oplus}) \) is a Banach space.

Let us next define the space of functions arising as left-hand sides of (4.1, which is the Fourier algebra.

**Definition 4.2.** The **Fourier algebra** of \( G \) is defined as

\[ A(G) := L^2(G)^* L^2(G)^* = \{f * g^* : f, g \in L^2(G)\}. \]

Endowed with the norm

\[ \|u\|_{A(G)} = \inf\{\|f\|_2\|g\|_2 : u = f * g^*\} \]

\( A(G) \) becomes a Banach space of \( C_0 \)-functions, with \( \|u\|_{A(G)} \geq \|u\|_\infty \). \( A(G) \) is closed with respect to pointwise multiplication and conjugation, which makes \( A(G) \) a Banach-*-algebra.

**Remark 4.3.** (a) By definition, \( A(G) \) is just the space of coefficient functions for \( \lambda_G \) and its subrepresentations. Hence it seems a natural object of study in connection with continuous wavelet transforms. However, neither the norm on \( A(G) \) nor its algebra structure seem to be related in a canonical way to questions concerning admissible vectors and wavelet transforms. Hence the usefulness of \( A(G) \) in this connection is dubious. For our purposes, focussing on the Plancherel transform and its inversion, the benefit of considering \( A(G) \) mainly lies in clarifying the role of the von Neumann algebra \( VN_l(G) \) in connection with inversion formulae, and in the notion of positivity which will be useful for convergence issues.
(b) Note that 4.2 is not the initial definition of $A(G)$ given in [41], but equivalent to the original definition because of [41, Théorème, p. 218]. Likewise, the norm given here is not the original definition in [41], but it coincides with it by [41, Lemme 2.14].

(c) If $u \in A(G)$, so are $u^*, \overline{u}$ and $\check{u}$, the latter defined as $\check{u}(x) = u(x^{-1})$; see [41, Proposition 3.8]. Moreover, we have

$$\|u^*\|_{A(G)} = \|\overline{u}\|_{A(G)} = \|\check{u}\|_{A(G)}.$$ 

The following theorem was given in [81] for unimodular groups.

**Theorem 4.4.** Let $A = (A_\sigma)_{\sigma \in \hat{G}} \in B_1^\oplus$. Then

$$a(x) = \mathcal{F}_{A(G)}^{-1}(A)(x) = \int_{\hat{G}} \text{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma)$$

defines a function $a \in A(G)$. It satisfies the Parseval equality

$$\int_G f(x) a(x) dx = \int_{\hat{G}} \text{trace}(\sigma(f) A_\sigma^*) d\nu_G(\sigma),$$

for all $f \in L^1(G)$. The linear operator $\mathcal{F}_{A(G)}^{-1} : B_1^\oplus \to A(G)$ is onto.

**Proof.** Let $A_\sigma = U_\sigma |A_\sigma|$ be the polar decomposition of $A_\sigma$. Then $B_{1,\sigma} = U_\sigma |A_\sigma|^{1/2}$ and $B_{2,\sigma} = |A_\sigma|^{1/2}$ defines elements $B_{1,\sigma}, B_{2,\sigma} \in B_2^\oplus$, since

$$\|B_1\|_{B_2^\oplus}^2 = \int_{\hat{G}} \text{trace}(B_{1,\sigma} B_{1,\sigma}^*) d\nu_G(\sigma)$$

$$= \int_{\hat{G}} \text{trace}(U_\sigma^* U_\sigma |A_\sigma|) d\nu_G(\sigma)$$

$$= \|A\|_{B_1^\oplus},$$

and similarly $\|B_2\|_{B_2^\oplus}^2 = \|A\|_{B_1^\oplus}$. For measurability confer Lemma 3.7. Denoting by $b_1, b_2 \in L^2(G)$ the respective preimages under the Plancherel transform, we find

$$a(x) = \int_{\hat{G}} \text{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma)$$

$$= \int_{\hat{G}} \text{trace}(B_{1,\sigma} B_{2,\sigma} \sigma(x)^*) d\nu_G(\sigma)$$

$$= (b_1, \lambda_G(x)b_2)$$

$$= (b_1 \ast b_2^*)(x),$$

hence $a \in A(G)$. If, conversely, $a = b \ast b_2^* \in A(G)$, define $(B_{1,\sigma})_{\sigma \in \hat{G}} = \hat{b}_1$. If we then let $A_\sigma = B_{1,\sigma} B_{2,\sigma}^*$, the same calculation shows that $a = \mathcal{F}_{A(G)}^{-1}(A)$. Hence the mapping is onto.
Finally, let us show the Parseval equality. Let \( A_{\sigma} = B_{1,\sigma}B_{2,\sigma} \) as above, and denote by \( h_1, h_2 \) the inverse Plancherel transform of \((B_{2,\sigma}^*,\sigma)_{\sigma \in \hat{G}}\) and \((B_{1,\sigma})_{\sigma \in \hat{G}}\), respectively. It follows that

\[
\int_{\hat{G}} \text{trace}(\sigma(f)A_{\sigma}^*)d\nu_G(\sigma) = \int_{\hat{G}} \text{trace}(\sigma(f)B_{2,\sigma}^*B_{1,\sigma}^*)d\nu_G(\sigma) \\
= \int_{\hat{G}} \text{trace}((f * h_1)\check{}(\sigma)B_{1,\sigma}^*)d\nu_G(\sigma) \\
= \int_{\hat{G}} (f * h_1)(x)\overline{h_2(x)}dx \\
= \int_{\hat{G}} f(y) \int_{\hat{G}} h_1(y^{-1}x)\overline{h_2(x)}dxdy \\
= \int_{\hat{G}} f(y) \int_{\hat{G}} h_2(x)h_1(y^{-1}x)dx dy \\
= \int_{\hat{G}} f(y) \int_{\hat{G}} \text{trace}(B_{1,\sigma}B_{2,\sigma}^*\sigma(y)^*)d\nu_G(\sigma) dy \\
= \int_{\hat{G}} f(y)a(y)dy .
\]

**Remark 4.5.** Below we will show that the Fourier inversion formula maps \( B_1^{\oplus} \) isometrically onto \( A(G) \). Now we are faced with the somewhat puzzling situation that on the one hand, Plancherel measure – which defines the norm on \( B_1^{\oplus} \) – is not uniquely given, whereas the norm on \( A(G) \) is defined independently of a choice of Plancherel measure. This apparent contradiction is easily resolved: Suppose that \( \nu_1 \) and \( \nu_2 \) are two different choices of Plancherel measure. Denote the corresponding spaces of integrable trace class fields by \( B_1^{\oplus}(\nu_i) \). Then there exists a canonical isometric isomorphism \( T : B_1^{\oplus}(\nu_1) \rightarrow B_1^{\oplus}(\nu_2) \), given by pointwise multiplication with \( \frac{d\nu_1}{d\nu_2} \). Moreover, if we denote the corresponding Fourier inversion operators by

\[
\mathcal{F}_i^{-1} : B_1^{\oplus}(\nu_i) \rightarrow A(G) ,
\]

an easy computation establishes that \( \mathcal{F}_2^{-1} = \mathcal{F}_1^{-1} \circ T \).

It is similarly remarkable that the Duflo-Moore operators do not make a single appearance in this section.

A most useful feature of \( A(G) \) is its close relationship to the left von Neumann algebra \( VN_l(G) \), as witnessed by the next theorem [41, Théorème 3.10]:

**Theorem 4.6.** Let \( A(G)' \) denote the Banach space dual of \( A(G) \). For all \( T \in VN_l(G) \), there exists a unique linear functional \( \varphi_T \in A(G)' \) such that,

\[
\varphi_T((f * g)^\vee) = \langle Tf, g \rangle .
\]
Here $\vee$ denotes the involution from Remark 4.3(c). The mapping $T \mapsto \varphi_T$ is an isometric isomorphism $VN_l(G) \to A(G)'$, which is bicontinuous if $VN_l(G)$ is equipped with the ultra-weak topology and $A(G)'$ with the weak$^*$-topology $\sigma(A(G)', A(G))$. Conversely, the ultra-weakly continuous linear forms on $VN_l(G)$ are precisely given by the mappings $T \mapsto \varphi_T(u)$, for fixed $u \in A(G)$.

**Definition 4.7.** The predual property of $A(G)$ allows to lift the action of $VN_l(G)$ on itself to an action on $A(G)$, via duality [41, 3.16]. More precisely, for $T \in VN_l(G)$, let $T \mapsto \tilde{T}$ denote the adjoint of the involution $a \mapsto \tilde{a}$. Given $u \in A(G)$, the functional

$$VN_l(G) \ni S \mapsto \varphi_{\tilde{T}S}(u)$$

is ultraweakly continuous, hence corresponds to a unique element $Tu \in A(G)$.

For $u \in A(G) \cap L^2(G)$ and $T \in VN_l(G)$ the notation $Tu$ is ambiguous, but [41, Proposition 3.17] notes that the two possible meanings coincide, and that $u \mapsto Tu$ is in fact norm-continuous on $A(G)$. Hence an alternative way of defining the mapping $u \mapsto Tu$ could proceed by extending it by continuity from $A(G) \cap L^2(G)$, which by [41, Proposition 3.4] is dense in $A(G)$.

Positivity, as defined next, will be useful in connection with convergence issues.

**Definition 4.8.** A function $u$ on $G$ is called of **positive type** if for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in G$ the matrix $(u(x_i^{-1}x_j))_{i,j=1,\ldots,n}$ is positive semi-definite. In such a case we write $u \gg 0$. If $u_1, u_2 \gg 0$ with $u_1 - u_2 \gg 0$, we write $u_1 \gg u_2$.

**Remark 4.9.** $u \gg 0$ implies $u(x) = \overline{u(x^{-1})}$, i.e. $u = u^*$. Moreover, it is obvious that $u \gg 0$ iff $\overline{u} \gg 0$. Particular examples of functions of positive type are

$$x \mapsto \langle \pi(x)\xi, \xi \rangle, \quad x \mapsto \langle \xi, \pi(x)\xi \rangle,$$

where $\pi$ is an arbitrary unitary representation, and $\xi \in \mathcal{H}_\pi$. For the first example, confer [35, 13.4.5], while the second is just the complex conjugate of the first. The first example is the typical way of referring to functions of positive type, while the second is more adapted to coefficient functions.

Taking $\pi = \lambda_G$, we find in particular for all $f \in L^2(G)$ that $f * f^* \gg 0$. Conversely, [57, p.73, Théorème 17] states that for all $u \in A(G)$ with $u \gg 0$ there exists $f \in L^2(G)$ such that $u = f * f^*$.

Next let us take a closer look at the duality between $VN_l(G)$ and $A(G)$ on the Fourier side. For this purpose we need an invariance property of Plancherel measure under the taking contragredients. The result is probably folklore; it is mentioned for instance in [81, Section 3]. We have not been able to locate a proof, though. If $\pi$, $\sigma$ are two irreducible representations with $\pi \simeq \sigma$, we
use the unique unitary intertwining operator $\mathcal{H}_\pi \to \mathcal{H}_\sigma$ to identify $\pi$ with the standard realization of $\bar{\sigma}$. This needs to be kept in mind in the next two lemmas, where we do not explicitly distinguish between operators on $\mathcal{H}_\pi$ and operators on $\mathcal{H}_{\bar{\sigma}}$, even though for a given measurable realization of $\bar{G}$ the representative of $[\bar{\sigma}]$ will not necessarily be the standard realization of the representative of $[\sigma]$.

**Lemma 4.10.** (a) If $G$ is unimodular and $\nu_G$ is the canonical choice of Plancherel measure, then $\nu_G$ is invariant under the mapping $\sigma \mapsto \bar{\sigma}$.

(b) If $G$ is nonunimodular, and $\nu_G$ is constructed from the canonical Plancherel measure of $\text{Ker}(\Delta_G)$ according to Proposition 3.50, then $\nu_G$ is invariant under the mapping $\sigma \mapsto \bar{\sigma}$.

**Proof.** For part (a) we let $\bar{\nu}$ denote the measure given by $\bar{\nu}(A) = \nu_G([\sigma : \sigma \in A])$. We first show that $\bar{\nu}$ is $\nu_G$-absolutely continuous. For this purpose observe that $\lambda_G \times \varrho_G \simeq \int_G \sigma \otimes \bar{\sigma} d\nu_G(\sigma)$. Now the fact that taking contragredients commutes with taking direct integrals and tensor products shows that $\Lambda_G \simeq \varrho_G$. On the other hand, $\lambda_G \simeq \varrho_G$ via the involution $f \mapsto \Delta_G^{-1/2} f^*$. Hence $\lambda_G \simeq \bar{\lambda}_G$. Now $\pi = \int_G \sigma d\nu_G(\sigma)$ is multiplicity-free and quasi-equivalent to $\lambda_G$, while $\bar{\pi} = \int_G \sigma d\bar{\nu}_G(\sigma)$ is multiplicity-free and quasi-equivalent to $\varrho_G \simeq \lambda_G$. Hence $\pi \simeq \bar{\pi}$, which by [35, 5.4.6] entails that $\pi \simeq \bar{\pi}$. But then $\nu_G$ and $\bar{\nu}$ are equivalent.

In addition, we obtain that the unique unitary operators $\mathcal{H}_\sigma \mapsto \mathcal{H}_{\bar{\sigma}}$ intertwining the realization of $\bar{\sigma}$ used in the Plancherel decomposition with the standard realization on $\mathcal{H}_\sigma$, constitute a measurable field of operators (outside a $\nu_G$-nullset); confer [36, Theorem 4, p.238]. Hence the identification of operators on $\mathcal{H}_\sigma$ and $\mathcal{H}_{\bar{\sigma}}$ can be obtained by a measurable field of operators.

As a first consequence, we see that the operator

$$(A_\sigma)_{\sigma \in \hat{G}} \mapsto (A_{\bar{\sigma}})_{\bar{\sigma} \in \hat{G}}$$

defined for all $A \in \mathcal{B}_2^\oplus$ for which the right hand side is in $\mathcal{B}_2^\oplus$, is densely defined and closed: It factors into the unitary map

$$T : (A_\sigma)_{\sigma \in \hat{G}} \mapsto \left( A_{\bar{\sigma}} \sqrt{\frac{d\sigma}{d\nu_G}}(\sigma) \right)_{\sigma \in \hat{G}}$$

followed by the densely defined closed operator consisting in pointwise multiplication with $\sqrt{\frac{d\nu_G}{d\sigma}}$.

Next we recall that the canonical choice of Plancherel measure entails for $f \in L^1(G) \cap L^2(G)$ that $\hat{f}(\sigma) = \sigma(f)$. A simple computation establishes

$$T(\hat{f})(\sigma) = \bar{\sigma}(f) = \bar{\bar{\sigma}}(f) = \bar{\hat{f}}(\sigma) \quad (4.3)$$

Note that this entails in particular that $\mathcal{P}(L^1(G) \cap L^2(G)) \subset \text{dom}(T)$. Since taking conjugates of $L^2$-functions and Hilbert-Schmidt operators are obviously isometric operations, we obtain that
is a unitary operator on $\mathcal{B}^\oplus_2$, and $T$ is a densely defined closed operator coinciding with it on a dense subspace. Hence $T$ is unitary itself. But then the multiplication operator arising from the Radon-Nikodym derivative is unitary also, which entails $\frac{d\sigma}{d\nu_G} \equiv 1$.

This proves the statement concerning unimodular $G$. The statement for nonunimodular groups follows from this, the fact that $\text{Ind}_N^G \sigma \simeq \text{Ind}_N^G \overline{\sigma}$ [86, Theorem 5.1], and the construction of $\nu_G$ from a measure decomposition of $\nu_{\text{Ker}(\Delta_G)}$, as sketched in Section 3.8.

**Lemma 4.11.** Let $A \in \mathcal{B}^\oplus_1$, and $a = \mathcal{F}^{-1}_{A(G)}(A)$. Let $T \simeq (\hat{T}_\sigma \otimes 1)_{\sigma \in \hat{G}} \in VN_l(G)$. Then

$$\varphi_T(a) = \int_{\hat{G}} \text{trace}(\hat{T}_\sigma A_{\overline{\sigma}})d\nu_G(\sigma) .$$  \hspace{1cm} (4.4)

**Proof.** In view of 4.10 we may assume that the Plancherel measure is invariant under taking contragredients. Hence, if $(A_{\sigma})_{\sigma \in \hat{G}} \in \mathcal{B}^\oplus_1$, then $(A_{\overline{\sigma}})_{\sigma \in \hat{G}} \in \mathcal{B}^\oplus_1$ as well. In particular, for fixed $A$ the right hand side of (4.4) is ultraweakly continuous as a function of $T$.

By Theorem 4.6, the mapping $T \mapsto \varphi_T(a)$ is ultraweakly continuous as well, hence it remains to check (4.4) for $T = \lambda_G(x), x \in G$ (these operators span a dense subalgebra). But here we have by [41, 3.14 Remarque], that

$$\varphi_T(a) = a(x)$$

$$= \int_{\hat{G}} \text{trace}(A_{\sigma} \sigma(x)^*)d\nu_G(\sigma)$$

$$= \int_{\hat{G}} \text{trace}(A_{\overline{\sigma}}(x)^*)d\nu_G(\sigma)$$

$$= \int_{\hat{G}} \text{trace}(A_{\overline{\sigma}}(x))d\nu_G(x)$$

$$= \int_{\hat{G}} \text{trace}(\sigma(x)A_{\overline{\sigma}})d\nu_G(x) .$$

Here the penultimate equality used the relations $\overline{\sigma(x)^*} = \sigma(x)^t$ and trace $(ST^t) = \text{trace}(S^tT)$. Since $\lambda_G(x) \simeq (\sigma(x) \otimes 1)_{\sigma \in \hat{G}}$, we are done.

Now we can establish the Fourier transform on $A(G)$. For unimodular groups, the theorem is [81, Theorem 3.1], for nonunimodular groups it is new.

**Theorem 4.12.** (a) $\mathcal{F}^{-1}_{A(G)}$, as defined in Theorem 4.4, is one-to-one. The inverse operator

$$\mathcal{F}_{A(G)} : A(G) \to \mathcal{B}^\oplus_1$$

is thus a bijection, in fact an isometry of Banach spaces.
(b) If \( T \simeq (\widehat{T}_\sigma \otimes 1)_{\sigma \in \hat{G}} \in VN_l(G) \) and \( a \in A(G) \), then

\[
F_{A(G)}(Ta) = (\widehat{T}_\sigma F_{A(G)}(a))(\sigma)_{\sigma \in \hat{G}}.
\]

(c) Let \( a = F_{A(G)}^{-1}(A) \). Then

\[
a \gg 0 \iff A_\sigma \geq 0 \quad (\nu_G - \text{a.e.})
\]

Proof. Define \( a = F_{A(G)}^{-1}(A) \), and let \( A_\sigma = U_\sigma|A_\sigma| \) denote the polar decomposition. Define \( B_{1,\sigma} \) and \( B_{2,\sigma} \) as in the proof of 4.4. We recall that \( \|B_1\|_{B_2^\oplus} = \|A\|^{1/2} \), and \( F_{A(G)}^{-1}(A) = b_1 \ast b_2^* \). Then \( a = b_1 \ast b_2^* \) implies that

\[
\|f\|_{A(G)} \leq \|b_1\|_2 \|b_2\|_2 = \|A\|_{B_1^\oplus}.
\]

For the converse direction define \( T \in VN_l(G) \) as \( \widehat{T}_\sigma = (U_\sigma^* t \otimes 1 \). Then the previous lemma entails

\[
\varphi_T(a) = \int_{\hat{G}} \text{trace}(U_\sigma U_\sigma^* A_\sigma) d\nu_G(\sigma) = \|A\|_{B_1^\oplus}.
\]

But \( a \mapsto \varphi_\bullet(a) \) is an isometric embedding of \( A(G) \) into \( VN_l(G)' \), by 4.6. In addition, \( \|T\|_\infty \leq 1 \), being defined by a field of partial isometries. Hence \( \|a\|_{A(G)} \geq \|A\|_{B_1^\oplus} \), and \( F_{A(G)}^{-1} \) is shown to be isometric, in particular one-to-one. This closes the proof of (a).

For the proof of (b) fix \( T \in VN_l(G) \), and denote the corresponding operator field on the Plancherel transform side by \( (\widehat{T}_\sigma \otimes 1)_{\sigma \in \hat{G}} \). Consider the mappings

1. \( a \mapsto Ta \)
2. \( a \mapsto F_{A(G)}^{-1}(A) \left( (\widehat{T}_\sigma F_{A(G)}(a))(\sigma)_{\sigma \in \hat{G}} \right) \)

on \( A(G) \). We claim that they coincide on the subspace \( C_c(G) \ast C_c(G) \subset A(G) \), which is dense by [41, Proposition 3.4]. Indeed, let \( a = f \ast g^* \) with \( f, g \in C_c(G) \).

The usual calculation shows that \( F_{A(G)}(a) = (\widehat{f}(\sigma)\widehat{g}(\sigma)^*)_{\sigma \in \hat{G}} \). \( g \) is a bounded vector by 2.19, i.e., \( V_g \in VN_{l_r}(G) \). Since \( VN_l(G) \) and \( VN_{l_r}(G) \) commute,

\[
[T(a)](x) = [T(V_g f)](x) = [V_g(T(f))](x) = [T(f) \ast g^*](x)
\]

\[
= \int_{\hat{G}} \text{trace} \left( \widehat{T}_\sigma \widehat{f}(\sigma)\widehat{g}(\sigma)^* \sigma(x)^* \right) d\nu_G(\sigma),
\]

which shows the claim. Now mapping 1. is bounded with respect to \( \| \cdot \|_{A(G)} \) by [41, Proposition 3.17]. On the other hand, since \( F_{A(G)} \) is isometric, we only need to show that \( (\widehat{T}_\sigma \otimes 1)_{\sigma \in \hat{G}} \) acts as a bounded operator on \( B_1^\oplus \), which follows trivially from the boundedness of \( T \). Hence we have shown (b) on all of \( A(G) \).
For (c) first assume \( A_\sigma > 0 \), \( \nu_G \)-almost everywhere. Then taking \( \hat{g}_\sigma = \frac{A_{1/2}}{\sigma} \) defines \( \hat{g} \in \mathcal{B}_2 \), and the calculation

\[
\mathcal{F}^{-1}_{A(G)}(A)(x) = \int_G \text{trace}(\hat{g}_\sigma \hat{g}_\sigma^*(x)^*)d\nu_G(\sigma) = g * g^*(x)
\]

shows that indeed \( \mathcal{F}^{-1}_{A(G)}(A) \gg 0 \). Conversely, if \( a = \mathcal{F}^{-1}_{A(G)}(A) \gg 0 \), then \( a = g * g^* \), and a similar calculation establishes that \( \mathcal{F}_{A(G)}(a) = (\hat{g}(\sigma)\hat{g}(\sigma)^*)_{\sigma \in \hat{G}} \), which clearly is a field of positive operators.

The following lemma will be instrumental in establishing the Plancherel inversion formula.

**Lemma 4.13.** Let \( G \) be unimodular. Suppose that \( u_1, u_2 \in L^2(G) \cap A(G) \), with \( u_2 \gg u_1 \gg 0 \), and that \( u_1 \) is a bounded vector. Then \( \|u_2\|_2 \geq \|u_1\|_2 \).

**Proof.** It suffices to show that

\[
\langle u_1, u_2 \rangle \geq \|u_1\|_2^2,
\]

since this implies

\[
\|u_2\|_2^2 - \|u_1\|_2^2 \geq \|u_2\|_2^2 + \|u_1\|_2^2 - 2\langle u_1, u_2 \rangle = \|u_2 - u_1\|_2^2.
\]

By assumption, there exists \( \psi \in L^2(G) \) such that \( u_2 - u_1 = \psi * \psi \). Pick \( (\varphi_n)_{n \in \mathbb{N}} \subset C_c(G) \) converging to \( \psi \) in \( \|\cdot\|_2 \). Then

\[
|\langle \psi, u_1 * \psi \rangle - \langle \varphi_n, u_1 * \varphi_n \rangle| \leq |\langle \psi - \varphi_n, u_1 * \psi \rangle| + |\langle \varphi_n, u_1 * (\varphi_n - \psi) \rangle| \\
\leq \|\psi - \varphi_n\|_2 \|u_1 * \psi\|_2 + \|\varphi_n\|_2 \|u_1 * (\varphi_n - \psi)\|_2 \to 0
\]

using the boundedness of \( u_1 \). On the other hand, \( \langle \varphi_n * \varphi_n^*, u_1 \rangle \geq 0 \) by positivity of \( u_1 \) (\[35, 13.4.4\], observe that for unimodular groups our involution coincides with the one in \[35\]). Hence

\[
\langle u_2 - u_1, u_1 \rangle = \lim_{n \to \infty} \langle \varphi_n * \varphi_n^*, u_1 \rangle \geq 0,
\]

which finishes the proof.

### 4.2 Plancherel Inversion*

In this section we discuss the \( L^2 \)-functions which can be obtained by the Fourier inversion formula (4.1). Clearly these functions are precisely the intersection \( A(G) \cap L^2(G) \). However, the description on the Plancherel transform side is much less obvious. In view of the last section, any theorem describing pointwise Plancherel inversion via (4.1) will contain some statement on the relationship between the spaces \( \mathcal{B}_1^\oplus \) and \( \mathcal{B}_2^\oplus \) and the operators \( \mathcal{P} \) and \( \mathcal{F}_{A(G)} \).
A first conjecture, which turns out to be correct for unimodular groups, could be that \( F_{A(G)}^{-1} \) maps \( B_2^\oplus \cap B_1^\oplus \) bijectively onto \( L^2(G) \cap A(G) \). This statement implies in particular that \( F_{A(G)} \) and \( P \) coincide on \( L^2(G) \cap A(G) \).

In the nonunimodular case however, this cannot be expected. Using the next lemma, it is possible to compute at least for \( V_g f, f \in L^2(G), g \in C_c(G) \), what the image under Plancherel transform is:

\[
(V_g f) \hat{\pi} \sigma = \hat{f} \pi \sigma (\Delta_G^{-1/2} g^*) = [\hat{f} \pi \sigma g^* C \sigma].
\]

Recalling that operator fields of the form \( (\hat{f} \pi \sigma g) \sigma \in \hat{G} \) are typical elements of \( B_1^\oplus \), this calculation motivates the conjecture that \( L^2(G) \cap A(G) \) is the image under \( F_{A(G)}^{-1} \) of the space

\[
\{(A \sigma) \sigma \in \hat{G} \in B_1^\oplus : ([A \sigma C \sigma]) \sigma \in \hat{G} \in B_2^\oplus \}.
\]

Theorem 4.15 below proves this conjecture. But first a few basic computations concerning the interplay between convolution with functions in \( C_c(G) \) and the Duflo-Moore operators. Not all of them are needed, but we include them for completeness.

**Lemma 4.14.** Let \( f \in C_c(G) \).

1. For \( \nu_G \)-almost every \( \sigma \), we have \( \hat{f} \pi \sigma = C \sigma^{-1} \sigma (\Delta_G^{-1/2} f^*) \). In particular the right hand side is everywhere defined and bounded.
2. For \( \nu_G \)-almost every \( \sigma \), we have

\[
[\sigma(f) C \sigma^{-1}] = C \sigma^{-1} \sigma (\Delta_G^{-1/2} f),
\]

in particular the right hand side is everywhere defined and bounded.
3. For all \( g \in L^2(G) \), we have

\[
(\hat{g} \ast \hat{f})(\sigma) = \hat{g}(\sigma) \sigma (\Delta_G^{-1/2} f) ,
\]

\[
(\hat{f} \ast \hat{g})(\sigma) = \sigma(f).
\]
4. For all \( g \in L^2(G) \), we have

\[
(\Delta_g^{-1/2} \sigma g^*) \hat{\pi} \sigma = \hat{g}(\sigma)^* .
\]

**Proof.** For part (i) we invoke [104, Theorem 13.2], to find that, since \( C \sigma^{-1} \) is selfadjoint and \( \sigma(f) \) is bounded, \( (\sigma(f) C \sigma^{-1})^* = C \sigma^{-1} \sigma(f)^* \). Moreover, since \( \hat{f} \sigma \) is bounded, the right hand side of the last equation is everywhere defined. We conclude the proof of (i) by computing

\[
\langle \sigma(\Delta_G^{-1/2} f^*) \phi, \eta \rangle = \int_G \langle \Delta_G^{-1} (x) \overline{f(x^{-1})} \sigma(x) \phi, \eta \rangle dx
\]

\[
= \int_G \langle \phi, f(x^{-1}) \sigma(x^{-1}) \eta \rangle \Delta_G(x)^{-1} dx
\]

\[
= \int_G \langle \phi, f(x) \sigma(x) \eta \rangle dx
\]

\[
= \langle \phi, \sigma(f) \eta \rangle .
\]
4.2 Plancherel Inversion

For (ii) we first note that by (i), applied to $\Delta^{-1/2}_G f \in C_c(G)$, the right hand side is bounded and everywhere defined. Moreover, the left-hand side is bounded since $f \in L^1(G) \cap L^2(G)$. It thus remains to show that the equality holds on the dense subspace $\text{dom}(C^{-1}_\sigma)$: For $\phi, \eta \in \text{dom}(C^{-1}_\sigma)$ the definition of the weak operator integral yields

$$\langle \phi, \sigma(f)C^{-1}_\sigma \eta \rangle = \int_G \langle \phi, \sigma(x)C^{-1}_\sigma \eta \rangle f(x) d\mu_G(x) = \int_G \langle \phi, \Delta_G(x)^{-1/2}C^{-1}_\sigma \sigma(x) \eta \rangle f(x) d\mu_G(x) = \int_G \langle C^{-1}_\sigma \phi, \sigma(x) \eta \rangle \Delta_G(x)^{-1/2} f(x) d\mu_G(x) = \langle C^{-1}_\sigma \phi, \sigma(\Delta^{-1/2}_G f) \eta \rangle = \langle \phi, C^{-1}_\sigma \sigma(\Delta^{-1/2}_G f) \eta \rangle,$$

where the second equality uses the semi-invariance relation (3.51), and the selfadjointness of $C^{-1}_\sigma$ was used on various occasions. This shows (ii).

Part (iii) is then immediate from (i) and (ii), at least for $g \in L^1(G) \cap L^2(G)$. It extends by continuity to all of $L^2(G)$. The left-hand sides are continuous operators, being convolution operators with $f \in C_c(G)$ (see 2.19(b)), and the right hand sides are continuous because of inequality (3.19).

For part (iv), we first observe that both sides of the equation are unitary mappings, hence it is enough to check the equality on $C_c(G)$. Here (i) and (ii) give $\widehat{g}(\sigma)^* = C^{-1}_\sigma \sigma(\Delta^{-1/2}_G g^*) = \sigma(\Delta^{-1/2}_G g^*)C^{-1}_\sigma = (\Delta^{-1/2}_G g)^*(\sigma)$.

The following theorem is one of the central new results of this book. For unimodular groups, it is stated in [81, Corollary 2.4, Corollary 2.5], though we will point out below that the argument in [81] seems to contain a gap. In any case, the nonunimodular part is new; one direction was proved in [4].

**Theorem 4.15.** Let $(A_\sigma)_{\sigma \in \widehat{G}} \in B_1^{\oplus}$, and define

$$a(x) = \int_{\widehat{G}} \text{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma). \quad (4.5)$$

Then $a \in L^2(G)$ iff $([A_\sigma C_\sigma])_{\sigma \in \widehat{G}} \in B_2^{\oplus}$. In that case we have $([A_\sigma C_\sigma])_{\sigma \in \widehat{G}} = \widehat{a}$.

**Proof.** First assume that $([A_\sigma C_\sigma])_{\sigma \in \widehat{G}} \in B_2^{\oplus}$, and let $b$ be the inverse image of that under the Plancherel transform. Then for any $g \in L^1(G) \cap L^2(G)$,
where the last equation is due to (4.2). Now Lemma 2.17 yields $b = a$ almost everywhere, and thus $\hat{a} = (\lfloor A_{\sigma}C_{\sigma} \rfloor)_{\sigma} \in \hat{G}$.

Let us now show the converse direction. Assume that $a \in L^2(G)$. Let $A_{\sigma} = U_\sigma |A_{\sigma}|$ be the polar decomposition of $A_{\sigma}$, let $U \simeq (U_\sigma \otimes 1)_{\sigma} \in \hat{G}$ the element in $VN_l(G)$ corresponding to the partial isometries, and let

$$h(x) = \int_{\hat{G}} \text{trace}(\lfloor A_{\sigma} | \sigma(x) \rfloor^*) d\nu_G(\sigma).$$

By Theorem 4.12(b) we have that $h = U^* a$, where the right hand side denotes the action of $VN_l(G)$ on $A(G)$. But $a \in L^2(G)$ then implies $h \in L^2(G)$, by [41, 3.17]. Since $(A_{\sigma})_{\sigma} \in B_2^\oplus$ iff $(\lfloor A_{\sigma} \rfloor)_{\sigma} \in B_2^\oplus$, we may thus assume w.l.o.g. that $A_{\sigma} \geq 0$.

Now assume $G$ to be unimodular. Pick an increasing sequence of Borel sets $\Sigma_n \subset \hat{G}$ with $\|A_{\sigma}\|_2 \leq n$ on $\Sigma_n$, $\nu_G(\Sigma_n) < \infty$ and $\bigcup_{n \in \mathbb{N}} \Sigma_n = \hat{G}$ (up to a Plancherel nullset). Consider the fields $A^n = (1_{\Sigma_n}(\sigma)A_{\sigma})_{\sigma} \in \hat{G}^\oplus$. Then the direction proved first shows that

$$a_n(x) = \int_{\hat{G}} \text{trace}(A^n_{\sigma} \sigma(x)^*) d\nu_G(\sigma)$$

defines a sequence in $L^2(G)$, with $\hat{a}_n = (A^n_{\sigma})_{\sigma} \in \hat{G}$. Observe that

$$(a - a_n)(x) = \int_{\hat{G} \setminus \Sigma_n} \text{trace}(A_{\sigma} \sigma(x)^*) d\nu_G(\sigma),$$

whence Theorem 4.12 (c) yields $a \gg a_n$. Next we check that the $a_n$ are bounded vectors: By construction, $\|A^n_{\sigma}\|_{\infty} \leq \|A^n_{\sigma}\|_2 \leq n$, hence for all $f \in L^2(G)$

$$V_{a_n} f(x) = \int_{\hat{G}} \text{trace}(\hat{f}(\sigma)A^n_{\sigma} \sigma^*(x)) d\nu_G(\sigma)$$

where $(\hat{f}(\sigma)A^n_{\sigma})_{\sigma} \in \hat{B}_2^\oplus$. Hence the previously established direction yields $V_{a_n} f \in L^2(G)$, i.e., $a_n$ is bounded. Now we may apply Lemma 4.13, implying $\|a_n\|_2 \leq \|a\|_2$. But then $\|A^n\|_{B_2^\oplus} \leq \|a\|_2$, and thus $A \in B_2^\oplus$.

Hence it remains to prove $a \in L^2(G) \Rightarrow (\lfloor A_{\sigma}C_{\sigma} \rfloor)_{\sigma} \in B_2^\oplus$ for $G$ nonunimodular. In the following we use the notations from Subsection 3.8.2. Recall
in particular, that \( \sigma = \text{Ind}_N^G \sigma_0 \), and that \( A_\sigma \) is given by a Hilbert-Schmidt valued kernel \((A_\sigma(\xi, \xi')))_{\xi, \xi' \in H}\). Again writing \( A_\sigma = F_\sigma G_\sigma \), with \( F_\sigma, G_\sigma \in \mathcal{B}_2^\oplus \), we appeal to 3.47 (a) and find that the integral kernel of \( A_\sigma \) is computed from the kernels of \( F_\sigma, G_\sigma \) by

\[
A_\sigma(\xi, \xi'') = \int_H F_\sigma(\xi, \xi') G_\sigma(\xi', \xi'') d\mu(\xi') ,
\]

whenever \( F_\sigma(\xi, \cdot), G_\sigma(\cdot, \xi'') \in L^2(H; \mathcal{B}_2(\mathcal{H}_\sigma)) \).

Lemma 3.47 (a) states that the right hand side is a Bochner integral converging in \( \mathcal{B}_1^\oplus(\mathcal{H}_\sigma_0) \).

We next compute the integral kernel of \( A_\sigma \sigma(g)^* \). Relation (3.56) implies for \( g = g_0 a(\gamma) \) and \( \sigma_0 = \tau(\sigma) \) that \( \sigma(g) A_\sigma^* \) has the integral kernel

\[
(\xi, \xi') \mapsto (\xi, \sigma_0)(g_0) \circ \sigma_0(A(\gamma, \xi)) \circ A_\sigma(\xi', \gamma^{-1}\xi)^* .
\]

Transposing yields the kernel

\[
B_{\sigma, \gamma} : (\xi, \xi') \mapsto A_\sigma(\xi, \gamma^{-1}\xi') \circ \sigma_0(A(\gamma, \xi'))^* \circ (\xi' . \sigma_0)(g_0)^* \]

for \( A_\sigma \sigma(g)^* \).

By Fubini’s theorem, the mapping \( a(\cdot, \gamma) : g_0 \mapsto a(g_0, \gamma) \) is in \( L^2(N) \), for almost all \( \gamma \). We intend to apply the unimodular part of the theorem to these functions. Plugging in the kernel for \( A_\sigma \sigma(g)^* \) and using the trace formula (3.40), we obtain

\[
a(g_0, \gamma) = \int \text{trace}(A_\sigma \sigma(x)^*) \ d\nu_G(\sigma)
= \int_{\hat{G}} \int_H \text{trace}(A_\sigma(\xi, \gamma^{-1}\xi) \sigma_0(A(\gamma, \xi))(\xi' . \sigma_0)(g_0)^*) \ d\xi d\nu_G(\sigma)
= \int_{\hat{G}} \text{trace}(A_\sigma(\xi, \gamma^{-1}\xi) \sigma_0(A(\gamma, \xi)) \Delta_G(\xi)^{-1} \sigma(g_0)^*) \ d\nu_N(\xi, \sigma_0) ,
\]

where the last equation uses (3.54). Next we estimate

\[
\int_{\hat{G}} \|A_\sigma(\xi, \gamma^{-1}\xi) \sigma_0(A(\gamma, \xi)) \Delta_G(\xi)^{-1}\|_1 d\nu_N(\xi, \sigma_0) = \int_{\hat{G}} \int_H \|A_\sigma(\xi, \gamma^{-1}\xi)\|_1 d\xi d\nu_G(\sigma)
\leq \int_{\hat{G}} \|F_\sigma\|_2 \|G_\sigma\|_2 d\nu_G(x) \leq \|F\|_{\mathcal{B}_2^\oplus} \|G\|_{\mathcal{B}_2^\oplus} ,
\]

where the first inequality is due to (3.41), and the second is again Cauchy-Schwarz. Hence we see that

\[
D^\gamma(\sigma_0, \xi) = A_\sigma(\xi, \gamma^{-1}\xi) \sigma_0(A(\gamma, \xi)) \Delta_G(\xi)^{-1} \ (\sigma_0 \in U_0, \xi \in H)
\]
defines a measurable operator field $D^\gamma \in B_2^\oplus(N)$, and that (4.6) is Fourier inversion applied to this field. Now for every $\gamma$ with $a(\cdot, \gamma) \in L^2(N)$ the unimodular part of the theorem implies that $D^\gamma \in B_2^\oplus(N)$, with

$$\|a(\cdot, \gamma)\|^2 = \int_{\hat{N}} \|D^\gamma(\sigma_0, \xi)\|^2 d\nu_N(\sigma_0, \xi).$$

But then Fubini’s theorem and (3.55) imply

$$\|a\|^2 = \int_H \int_{U_0} \int_H \|A_\sigma(\xi, \gamma^{-1}x)\|^2 \Delta_G(\gamma^{-1}) d\gamma d\xi d\nu_G(\sigma)$$

$$= \int_H \int_{\hat{G}} \int_H \|A_\sigma(\xi, \gamma^{-1}x)\|^2 \Delta_G(\gamma^{-1}) d\gamma d\xi d\nu_G(\sigma)$$

$$= \int_{\hat{G}} \int_H \int_H \|A_\sigma(\xi, \gamma^{-1}x)\|^2 \Delta_G(\gamma^{-1}) d\gamma d\xi d\nu_G(\sigma)$$

$$= \int_{\hat{G}} \int_H \int_H \|A_\sigma(\xi, \gamma)\Delta_G(\gamma^{-1}) d\gamma d\xi d\nu_G(\sigma) = \int_{\hat{G}} \int_H \int_H \|A_\sigma(\xi, \gamma)\Delta_G(\gamma) d\gamma d\xi d\nu_G(\sigma).$$

Hence we find in particular that for $\nu_G$-almost every $\sigma \in \hat{G}$ the operator with kernel

$$(\xi, \gamma) \mapsto A_\sigma(\xi, \gamma)\Delta_G(\gamma)^{-1/2}$$

is Hilbert-Schmidt. On the other hand, recalling that $C_\sigma$ acts via the multiplication with $\Delta_G(\gamma)^{-1/2}$, we see that $A_\sigma C_\sigma$ coincides with this Hilbert-Schmidt operator on $\text{dom}(C_\sigma)$. Hence $[A_\sigma C_\sigma]$ exists and is in $B_2(H_\sigma)$, and we conclude

$$\|a\|^2 = \int_G \|A_\sigma C_\sigma\|^2 d\nu_G(\sigma),$$

which finishes the proof.

**Remark 4.16.** (a) The unimodular version of the Plancherel inversion theorem was shown in [81], and the proof of

$$([A_\sigma C_\sigma])_{\sigma \in \hat{G}} \in B_2^\oplus \Rightarrow a \in L^2(G) \quad (4.7)$$

immediately carries over to the general setting. Note that the argument relies on Lemma 2.17.

This observation is crucial in connection with the proof for the converse direction

$$a \in L^2(G) \Rightarrow ([A_\sigma C_\sigma])_{\sigma \in \hat{G}} \in B_2^\oplus.$$

The argument given in [81] for unimodular groups uses (4.2) to establish
\[ \int_G a(x) \overline{k(x)} \, dx = \int_G \text{trace}(A_\sigma \hat{k}(\sigma)^*) \, d\nu_G(\sigma), \]

for all \( k \in L^1(G) \cap L^2(G) \), and then concludes by density of \( \mathcal{P}(L^1(G) \cap L^2(G)) \)
in \( B_2^\oplus \) that \( \hat{a} = A \). This is the mirror image of the argument for \( \{A_\sigma C_\sigma\}_{\sigma \in \hat{G}} \in B_2^\oplus \Rightarrow a \in L^2(G) \), with \( \mathcal{P}(L^1(G) \cap L^2(G)) \) replacing \( L^1(G) \cap L^2(G) \). But in Remark 2.18 we saw that density alone is insufficient, and an analogue of 2.17 for \( \mathcal{P}(L^1(G) \cap L^2(G)) \) instead of \( L^1(G) \cap L^2(G) \) does not seem easily available.

This is why the argument presented here is rather more complicated than the one given in [81]. Note in particular that we used the action of \( VN_i(G) \) on \( A(G) \) and the notion of positivity from Section 4.1. A substantially shorter argument could be provided if the following rather intuitive result from integration theory were true. Unfortunately I have not been able to prove it:

Let a sequence \( (a_n)_{n \in \mathbb{N}} \subset L^2(G) \) be given with \( a_n \to a \) uniformly. Suppose that the \( a_n \) have orthogonal increments. Then \( a \in L^2(G) \), with \( \|a\|_2 \geq \|a_n\|_2 \) for all \( n \).

Note however that this observation only allows to shorten the unimodular part of the proof.

(b) The implication (4.7), with a much more complicated proof, may be found in [4]. This direction allows to prove sufficient admissibility conditions, as shown in [53]. For necessity of these conditions however, the converse of (4.7) seems indispensable.

### 4.3 Admissibility Criteria

In this section we solve \( T1 \) through \( T3 \) for the general setting, and discuss the qualitative uncertainty property.

Since leftinvariant subspaces correspond to projections in \( VN_i(G) \), we obtain an answer to \( T1 \) as an immediate consequence of the Theorem 3.48 (c).

**Corollary 4.17.** Let \( \mathcal{H} \subset L^2(G) \) be a closed leftinvariant subspace. The projection \( P \) onto \( \mathcal{H} \) corresponds to a field of projections \( P \simeq (1 \otimes \hat{P}_\sigma)_{\sigma \in \hat{G}} \), i.e.

\[ (Pf)^*(\sigma) = \hat{f}(\sigma) \circ \hat{P}_\sigma. \]  

(4.8)

Next let us consider admissible vectors. The proof turns out to be largely analogous to the proof for the toy example, once the \( L^2 \)-convolution theorem is established. But this is now a formality.

**Theorem 4.18.** For \( f, g \in L^2(G) \) we have \( V_g f \in L^2(G) \) iff \( \{[\hat{f}(\sigma) \hat{g}(\sigma)^* C_\sigma]\}_{\sigma \in \hat{G}} \in B_2^\oplus \). In this case, we have \( \nu_G \)-almost everywhere

\[ (V_g f)^*(\sigma) = [\hat{f}(\sigma) \hat{g}(\sigma)^* C_\sigma]. \]  

(4.9)

In terms of operator domains, this means that

\[ \text{dom}(V_g) = \{ f \in L^2(G) : ([\hat{f}(\sigma) \hat{g}(\sigma)^* C_\sigma])_{\sigma \in \hat{G}} \in B_2^\oplus \}. \]
Proof. Apply Theorem 4.15 to $A_\sigma := \hat{f}(\sigma)\hat{g}(\sigma)^*$, and observe that the unitarity of Plancherel transform yields

$$(V_g f)(x) = \langle f, \lambda G(x) g \rangle = \int_G \text{tr}(A_\sigma(x)^*) d\nu G(x) .$$

Now we only need to note some technical facts concerning multiplication operators on tensor products, and the criteria for admissibility and boundedness can be derived.

Lemma 4.19. Let $\mathcal{H}, \mathcal{K}_1, \mathcal{K}_2$ be Hilbert spaces. Given a densely defined operator $A : \mathcal{K}_2 \to \mathcal{K}_1$, consider the operator $\mathcal{A} : \mathcal{H} \otimes \mathcal{K}_1 \to \mathcal{H} \otimes \mathcal{K}_2$ defined by letting $T \mapsto [TA]$, for all $T \in \mathcal{H} \otimes \mathcal{K}_1$ for which a bounded extension from $\text{dom}(A)$ exists and is in $\mathcal{H} \otimes \mathcal{K}_2$.

(a) $\mathcal{A}$ is closed.
(b) $\mathcal{A}$ is bounded iff $A$ has a bounded extension.
(c) Now assume $A = S^*C$, where $S \in \mathcal{H} \otimes \mathcal{K}_1$, and $C : \mathcal{H} \to \mathcal{H}$ is selfadjoint, with $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{H}$. If $S = \sum_{j \in J} a_j \otimes \varphi_j$, for some ONB $(\varphi_j)_{j \in J}$ of $\mathcal{K}$, then

$$\mathcal{A} \text{ is isometric } \iff (a_j)_{j \in J} \subset \text{dom}(C) \text{ and } (Ca_j)_{j \in J} \text{ is an ONS} .$$

(4.10)

Proof. For part (a) we pick a basis $(\eta_i)_{i \in I}$ of $\mathcal{H}_1$ to write arbitrary elements of $\mathcal{H} \otimes \mathcal{K}_1$ as

$$S = \sum_{i \in I} \eta_i \otimes b_i .$$

Then $\text{dom}(\mathcal{A})$ consists of all such $S$ for which $(b_i)_{i \in I} \subset \text{dom}(A^*)$ with $\sum_{i \in I} \|A^* b_i\|^2 < \infty$, and then

$$\mathcal{A} S = \sum_{i \in I} \eta_i \otimes A^* b_i .$$

Since $A^*$ is closed, it is easy to conclude from this that $\mathcal{A}$ is closed.

Moreover observe that the fact

$$\|\mathcal{A} S\|^2 = \sum_{i \in I} \|A^* b_i\|^2 ,$$

clearly entails that $\mathcal{A}$ is bounded iff $A^*$ is bounded, which entails (b).

For part (c) observe that the same argument proving (b) yields that $\mathcal{A}$ is isometric iff $A^* = CS$ is isometric. It is straightforward to conclude that $(a_j)_{j \in J} \subset \text{dom}(A)$, and

$$CS = \sum_{j \in J} (Ca_j) \otimes \varphi_j .$$

Since an isometry is characterized by the fact that the ONB $(\varphi_j)_{j \in J}$ must be mapped onto an ONS, $(Ca_j)_{j \in J}$ is an ONS iff $CS$ is isometric.
Compare the following theorem to Theorem 2.64. The sufficiency of the criteria may be found in [53]. For unimodular groups, the characterization of selfadjoint convolution operators on the Fourier side was proved by Carey [29].

**Theorem 4.20.** Let $\mathcal{H} \subset L^2(G)$ be a leftinvariant closed subspace, with orthogonal projection $P \simeq (1 \otimes \hat{P})_{\sigma \in \hat{G}}$. Then we have the following equivalences for $\eta \in \mathcal{H}$

- $\eta$ is admissible $\iff [C_\sigma \hat{\eta}(\sigma)]$ is an isometry on $\hat{P}_\sigma(\mathcal{H}_\sigma)$, (4.11)
- $\eta$ is bounded $\iff \sigma \mapsto \|[C_\sigma \hat{\eta}(\sigma)]\|_\infty$ is essentially bounded, (4.12)
- $\eta$ is cyclic $\iff \sigma \mapsto \|[C_\sigma \hat{\eta}(\sigma)]\|_\infty$ is injective on $\hat{P}_\sigma(\mathcal{H}_\sigma)$. (4.13)

Bounded vectors fulfill $\|V_\eta\|_\infty = \text{ess sup}_{\sigma \in \hat{G}} \|[C_\sigma \hat{\eta}(\sigma)]\|_\infty$. Moreover, $S \in L^2(G)$ is a selfadjoint convolution idempotent iff $[C_\sigma \hat{S}(\sigma)]$ is a projection, for $\nu_G$-almost every $\sigma$.

**Proof.** The chief technical difficulty remaining for the proof of necessity is that the $L^2$-convolution theorem for $V_\eta f$ holds only pointwise a.e. on the Plancherel transform side, and in principle the conull set may depend on $f$. Hence we need some additional functional analysis to make the argument work.

Pick a countable total subset $S \subset L^2(G)$. Then $\{\hat{f}(\sigma) : f \in S\}$ is total in $\mathcal{H}_\sigma$, for almost every $\sigma$; otherwise one could construct (measurably) the projections onto the nontrivial complements and obtain a nontrivial complement in $L^2(G)$. Suppose that $\eta$ is a bounded vector, say $\|V_\eta\|_\infty \leq k$. Then we find by Theorem 4.18 that for all $f \in S$

$$\int_{\hat{G}} \|[\hat{f}(\sigma) \hat{\eta}(\sigma)]^* C_\sigma\|_2^2 d\nu_G(\sigma) \leq k^2 \int_{\hat{G}} \|\hat{f}(\sigma)\|_2^2 d\nu_G(\sigma) .$$

Passing from $\hat{f}$ to $(\hat{f}(\sigma)1_B(\sigma))_{\sigma \in \hat{G}}$, for a Borel subset $B$, we see that we replace $\hat{G}$ as integration domain on both sides by $B$. Since $B$ can be arbitrary, this implies the inequality for the integrands, i.e.,

$$\|[\hat{f}(\sigma) \hat{g}(\sigma)^* C_\sigma]\|_2^2 \leq k^2 \|\hat{f}(\sigma)\|_2^2 .$$

Now the totality of $\{\hat{f}(\sigma) : f \in S\}$ shows that the inequality holds on a dense subspace of $\mathcal{H}_\sigma$, for almost all $\sigma \in \hat{G}$. Then parts (a) and (b) of Lemma 4.19 apply to yield that $\hat{\eta}(\sigma)^* C_\sigma$ is bounded, and norm $\leq k$ almost everywhere follows by density considerations.

The remaining necessary conditions in (4.12) and (4.11), as well as the norm equality

$$\|V_\eta\|_\infty = \text{ess sup}_{\sigma \in \hat{G}} \|[C_\sigma \hat{\eta}(\sigma)]\|_\infty .$$

(4.13) follow similarly. The converse directions are immediate from Theorem 4.18. Since $S$ is a selfadjoint convolution idempotent iff $V_S$ is a projection operator, the last statement is immediate.
Remark 4.21. The differences between the unimodular and the nonunimodular cases can be exemplified by the following observation: To any $f \in L^1(G) \cap L^2(G)$ we can associate at least three objects on the Fourier transform side: The Fourier transform $(\sigma(f))_{\sigma \in \hat{G}}$, the Plancherel transform $(\hat{f}(\sigma))_{\sigma \in \hat{G}}$, and the decomposition of the operator $V_g \simeq (1 \otimes \hat{T}_\sigma)_{\sigma \in \hat{G}}$. In the unimodular case, all three objects are basically identical: $\hat{f}(\sigma) = \sigma(f)$ and $\hat{T}_\sigma = \sigma(f)^*$. In the nonunimodular case however, they all differ by suitable powers of the Duflo-Moore operator.

Now we can easily characterize the subrepresentations of $\lambda_G$ with admissible vectors. The statement concerning unimodular groups is partly contained in [29, Theorem 2.10]; in this form the theorem appeared in [53]. A special case of the theorem for the reals was given in Theorem 2.65. Also, the discussion of direct sums of discrete series representations contained in Remark 2.32, in particular for unimodular groups, is a special case of this theorem.

Theorem 4.22. Let $\mathcal{H} \subset L^2(G)$ be a leftinvariant closed subspace, and let $P \simeq (1 \otimes \hat{P}_\sigma)_{\sigma \in \hat{G}}$ denote the projection onto $\mathcal{H}$. Then $\mathcal{H}$ has admissible vectors iff either

- $G$ is unimodular, almost all $\hat{P}_\sigma$ have finite rank and

$$\nu_\mathcal{H} = \int_{\hat{G}} \text{rank}(\hat{P}_\sigma) d\nu_G(\sigma) < \infty \ . \quad (4.14)$$

In this case every admissible vector $g \in \mathcal{H}$ fulfills $\|g\|^2 = \nu_\mathcal{H}$.  

- $G$ is nonunimodular. In that case, there exist admissible vectors with arbitrarily small or big norm.

Proof. Assume first that $G$ is unimodular. Suppose $g$ is an admissible vector for $\mathcal{H}$, and define $h = g^* * g$. Then we have $P = (V_g)^* \circ (V_g) = V_{g^*} \circ V_g = V_h$ and $h = V_{g^*} g \in L^2(G)$ (note that $V_{g^*}$ is a bounded operator on all of $L^2(G)$). Applying Theorem 4.18 first to $V_h$ and then to $V_{g^*}$ yields

$$P_\sigma = \hat{h}(\sigma)^* = \hat{g}(\sigma)^* \hat{g}(\sigma) \ ,$$

$\nu_G$-almost everywhere. Hence

$$\|g\|^2 = \int_{\hat{G}} \text{tr}(\hat{g}(\sigma)^* \hat{g}(\sigma)) d\nu_G(\sigma)$$

$$= \int_{\hat{G}} \text{tr}(\hat{g}(\sigma)^* \hat{g}(\sigma)) d\nu_G(\sigma)$$

$$= \int_{\hat{G}} \text{rank}(P_\sigma) d\nu_G(\sigma) \ .$$

For the sufficiency of (4.14) we note that $(\hat{P}_\sigma)_{\sigma \in \hat{G}} \in B^\oplus_2$, and we let $g$ be the inverse Plancherel transform of that. Then Theorem 4.18 shows that $P = V_g$, 


which means that $V_g$ is the identity operator on $\mathcal{H}$, and $g$ is admissible.
In the nonunimodular case, the statement follows from Theorem 4.23 and its corollary.

**Theorem 4.23.** Let $G$ be nonunimodular. Then there exists an operator field $(A_\sigma)_{\sigma \in \hat{G}} \in B_2^\oplus$ such that $[C_\sigma A_\sigma]$ is an isometry, for $\nu_G$-almost every $\sigma \in \hat{G}$. As a consequence, if $a \in L^2(G)$ is the preimage of $A$ under the Plancherel transform, $a$ is admissible for $\lambda_G$.

**Proof.** The proof uses the techniques and objects described in detail in Section 3.8.2, in particular the notions from Proposition 3.50. We first give the $A_\sigma$ pointwise and postpone the questions of measurability and square-integrability. Pick $c > 1$ in such a way that \( \{ \gamma \in H : 1 \leq \Delta_G^{-1/2}(\gamma) < c \} \) has positive Haar measure, and define, for $n \in \mathbb{N}$, $S_n := \{ \gamma \in H : c^n \leq \Delta_G^{-1/2}(\gamma) < c^{n+1} \}$. Let $(v_n^\sigma)_{n \in \mathbb{N}} \subset \mathcal{H}_\sigma = L^2(H, d\mu_H; \mathcal{H}_\tau(\sigma))$ be an ONB. Moreover let $(v_n^\sigma)_{n \in \mathbb{N}} \subset L^2(H, d\mu_H; \mathcal{H}_\tau(\sigma))$ be a sequence of unit vectors with $\text{supp}(v_n^\sigma) \subset S_n$. Define the operator $A_\sigma$ by

$$A_\sigma = \sum_{n \in \mathbb{N}} \| \Delta_G^{-1/2} v_n^\sigma \|^{-1} v_n^\sigma \otimes u_n^\sigma.$$ 

This defines a Hilbert-Schmidt operator, since $\| \Delta_G^{-1/2} v_n^\sigma \| \geq c^n$.

On the other hand, the construction of the $v_n^\sigma$ implies that $A_\sigma$ maps $\mathcal{H}_\sigma$ into $\text{dom}(C_\sigma)$. In fact, given any $f = \sum_{n \in \mathbb{N}} \langle f, u_n^\sigma \rangle u_n^\sigma$, the disjointness of the supports of the $v_n^\sigma$ implies for all $h \in H$ that

$$A_\sigma f(h) = \sum_{n \in \mathbb{N}} \| \Delta_G^{-1/2} v_n^\sigma \|^{-1} \langle f, u_n^\sigma \rangle v_n^\sigma(h)$$

$$= \begin{cases} \| \Delta_G^{-1/2} v_n^\sigma \|^{-1} \langle f, u_n^\sigma \rangle v_n^\sigma(h) : h \in S_n \text{ for some } n \\ 0 : \text{ otherwise} \end{cases}.$$ 

Thus

$$\| C_\sigma A_\sigma f \|^2 = \sum_{n \in \mathbb{N}} \int_{S_n} \| \Delta_G^{-1/2} v_n^\sigma \|^{-2} \left| \Delta_G^{-1/2}(h) \langle f, u_n^\sigma \rangle v_n^\sigma(h) \right|^2 dh$$

$$= \sum_{n \in \mathbb{N}} |\langle f, u_n^\sigma \rangle|^2 \| \Delta_G^{-1/2} v_n^\sigma \|^{-2} \int_{S_n} \| \Delta_G^{-1/2}(h) v_n^\sigma(h) \|^2 dh$$

$$= \| f \|^2.$$ 

Therefore $C_\sigma A_\sigma$ is an isometry.

Let us next address the measurability requirement. We have already constructed a measurable family $(u_n^\sigma)_{n \in \mathbb{N}}$ of ONB’s, so we only have to ensure that the images $(\| \Delta_G^{-1/2} v_n^\sigma \|^{-1} v_n^\sigma)_{n \in \mathbb{N}}$ can be chosen measurably as well. Pick any family $(b_n)_{n \in \mathbb{N}} \subset L^2(H)$ of unit vectors, such that $b_n$ is supported in $S_n$. 


Moreover, let \((\xi^\sigma)_{\sigma \in V}\) be a measurable field of unit vectors \(\xi^\sigma \in H_{\tau(\sigma)}\), and define \(v^\sigma_n = b_n \xi^\sigma\). Then

\[
\sigma \mapsto \langle \| \Delta_G^{-1/2} a_n \|^{-1} v^\sigma_n, a_n e_k^\sigma \rangle = \| \Delta_G^{-1/2} a_n \|^{-1} \langle b_n, a_n \rangle \langle \xi^\sigma, e_k^\sigma \rangle
\]

is measurable by the choice of the \(\xi^\sigma\).

Finally, let us provide for square-integrability. For this purpose we observe that we may assume the constant \(c\) picked above to be \(\geq 2\), and then \(\| A_\sigma \|_2^2 \leq 2\). Moreover, if we shift the construction in the sense that \(v^\sigma_n \mapsto \| \Delta_G^{-1/2} v^\sigma_{n+k} \|^{-1} v^\sigma_{n+k}\), for \(k > 0\), we obtain \(\| A_\sigma \|_2^2 < 2^{1-k}\), while preserving all the other properties of \(A_\sigma\). With this in mind, we can easily modify the construction to obtain an element of \(B_2^G\): Since \(G\) is separable, \(\nu_G\) is \(\sigma\)-finite, i.e., \(\hat{G} = \bigcup_{n \in \mathbb{N}} \Sigma_n\) with the \(\Sigma_n\) pairwise disjoint and \(\nu_G(\Sigma_n) < \infty\). Shifting on \(\Sigma_n\) by \(k_n \in \mathbb{N}\) with \(\nu_G(\Sigma_n) 2^{-k_n} < 2^{-n}\) ensures square-integrability without destroying measurability. (The latter is obvious on \(\Sigma_n\).) Hence we are done.

The shifting argument employed in the proof also provides the following fact.

**Corollary 4.24.** The operator field \((A_\sigma)_{\sigma \in \hat{G}}\) from Theorem 4.23 can be constructed with arbitrarily small or big norm.

The following corollary is obtained by combining Theorems 2.42 and 4.22. We expect that it has already been noted elsewhere; it can also be derived from the results in the paper by Arnal and Ludwig [14].

**Corollary 4.25.** Let \(G\) be a nondiscrete unimodular group. Then

\[
\int_{\hat{G}} \dim(H_\sigma) d\nu_G(\sigma) = \infty .
\]

For completeness, we note the following characterization of unimodularity, which follows from Theorems 2.42 and 4.22.

**Corollary 4.26.** Let \(G\) be a nondiscrete group. Then \(\lambda_G\) has an admissible vector iff \(G\) is nonunimodular.

Theorem 4.22 and Corollary 3.51 imply the following remarkable property.

**Corollary 4.27.** Suppose that \(G\) is a nonunimodular group. Let \(\pi\) be a representation of \(G\), and assume that \(\pi = \bigoplus_{n \in \mathbb{N}} \pi_n\). If each \(\pi_n\) has an admissible vector, so does \(\pi\).

The corollary can be applied to the results of Liu and Peng [83] to construct a continuous wavelet transform on the Heisenberg group \(H\). The authors considered a particular group extension \(H \rtimes \mathbb{R}\) and its action on \(L^2(H)\) for the construction of wavelet transforms. They established that this representation decomposes into an infinite direct sum of discrete series representations, and they gave admissibility conditions for each. Now the corollary provides the
existence of an admissible vector for the whole representation. Corollary 5.28 below shows that this type of construction in fact works for arbitrary homogeneous Lie groups.

We next give a generalization of the qualitative uncertainty principle to nonunimodular groups. For the unimodular version, confer \[14\].

**Corollary 4.28.** Suppose that $G$ has a noncompact connected component. For $f \in \mathcal{L}^2(G)$ let $P_\sigma$ denote the projection onto $\left( \text{Ker}(\hat{f}(\sigma)) \right)^\perp$, and assume that

$$ r(\sigma) = \|P_\sigma C_\sigma^{-1}\|^2 $$

is well-defined and finite $\nu_G$-almost everywhere. Note that in the unimodular case $r(\sigma) = \text{rank}(\hat{f}(\sigma))$. Suppose that $f$ fulfills

\begin{enumerate}[(i)]  \item $|\text{supp}(f)| < \infty$ and  \item $\int_{\hat{G}} r(\sigma) d\nu_G(\sigma) < \infty$. \end{enumerate}

Then $f = 0$.

**Proof.** Consider the operators $\hat{S}_\sigma = [C_\sigma^{-1} P_\sigma]$, by assumption $(\hat{S}_\sigma)_{\sigma \in \hat{G}} \in \mathcal{B}_2^\oplus$. If we let $S$ denote the inverse Plancherel transform of that, the $\mathcal{L}^2$-convolution Theorem entails that

$$ (V_S f)^\wedge = \left( [\hat{f}(\sigma) P_\sigma C_\sigma^{-1} C_\sigma] \right)_{\sigma \in \hat{G}} = \hat{f}, $$

since by construction $\hat{f}(\sigma) \circ P_\sigma = \hat{f}(\sigma)$. In other words, $f = V_S f = f \ast S^*$. Moreover, since $[C_\sigma \hat{S}_\sigma] = P_\sigma$ is a projection operator, $S = S^*$ by Theorem 4.20, and thus $f = f \ast S$. Now assumption (i) and Theorem 2.45 imply $f = 0$.

**Remark 4.29.** In the real case condition (ii) specializes to the well-known condition $|\text{supp}(\hat{f})| < \infty$; this is our excuse for calling Corollary 4.28 a qualitative uncertainty principle. In the unimodular case the analogy to the qualitative uncertainty property is still quite comprehensible: The Plancherel measure of the support of $f$ is simply weighted with the rank of $\hat{f}(\sigma)$. If we let $\mathcal{H}$ denote the leftinvariant closed subspace generated by $f$, then $\int_{\hat{G}} r(\sigma) d\nu_G(\sigma) = \nu_{\mathcal{H}}$, the latter being the constant encountered in Theorem 4.22.

For nonunimodular groups the quantity $\int_{\hat{G}} r(\sigma) d\nu_G(\sigma)$ is more difficult to interpret. It is however independent of the choice of Plancherel measure: If we pass to an alternative pair $(\tilde{\nu}_G, (\tilde{C}_\sigma)_{\sigma \in \hat{G}})$ of Plancherel measure and associated Duflo-Moore-operators and define $\tilde{r}$ using the $\tilde{C}_\sigma$, the renormalizations of $\tilde{\nu}_G$ and $\tilde{C}_\sigma$ cancel to yield

$$ \int_{\hat{G}} \tilde{r}(\sigma) d\tilde{\nu}_G(\sigma) = \int_{\hat{G}} r(\sigma) d\nu_G(\sigma), $$

Let us next consider $T4$, i.e., the question how to make the criteria derived for the regular representation applicable to arbitrary representations $\pi$. We intend to employ the existence and uniqueness of direct integral decompositions for this task: Decomposing $\pi$ into irreducibles, we can check containment in $\lambda_G$ by comparing the underlying measure to Plancherel measure, and comparing multiplicities (if necessary). Moreover, once containment is established in this way, the admissibility conditions for $\lambda_G$ directly carry over to the direct integral decomposition of $\pi$, much in the way that we could directly establish admissibility criteria for representations of $\mathbb{R}$, as described in Remark 2.70. Using Lemma 4.19(c), we can in addition break the admissibility condition formulated for operator fields down to conditions involving rank-one operators. Thus we obtain orthogonality conditions generalizing the discrete series case from Theorem 2.31.

**Remark 4.30.** Given an arbitrary representation $(\pi, \mathcal{H}_\pi)$ of a type I group, the following steps need to be carried out for establishing admissibility conditions:

- Explicitly construct a unitary intertwining operator
  \[ T : \mathcal{H}_\pi \to \int_\hat{G}^{\oplus} \mathcal{H}_{\pi}(\sigma) d\tilde{\nu}(\sigma) \]
  \[ \pi \simeq \int_\hat{G}^{\oplus} m_{\pi}(\sigma) \cdot \sigma d\tilde{\nu}(\sigma) \]
  where $\tilde{\nu}$ is a suitable measure on $\hat{G}$ and $m$ is a multiplicity function.

- There exist admissible vectors if and only if the following questions are answered in the affirmative:
  - Is $\tilde{\nu}$ $\nu_G$-absolutely continuous?
  - Is $m_{\pi}(\sigma) \leq \dim(\mathcal{H}_\sigma)$, $\tilde{\nu}$-almost everywhere?
  - If $G$ is unimodular, does the relation
  \[ \int_\hat{G} m_{\pi}(\sigma) d\nu_G(\sigma) < \infty \]
  hold?
  Note that if $G$ is nonunimodular, the answer to the second question is "yes" by Corollary 3.51.

- Compute the Radon-Nikodym derivative $\frac{d\tilde{\nu}}{d\nu_G}$.
  If $G$ is nonunimodular, compute the Duflo-Moore operators $C_\sigma$, using e.g.
  the description in Proposition 3.50, or the semi-invariance relation (3.51). $T$ maps vectors $\eta \in \mathcal{H}_\pi$ to measurable families $((T\eta)(\sigma, i))_{\sigma \in \hat{G}, i = 1, \ldots, m_{\pi}(\sigma)}$. Then $\eta$ is admissible iff it fulfills the following orthonormality relations, for $\nu_G$-almost every $\sigma \in \Sigma$:
  \[ \left( \frac{d\tilde{\nu}}{d\nu_G}(\sigma) \right)^{1/2} \| C_\sigma(T\eta)(\sigma, i) \| = 1, \text{ for } 1 \leq i \leq m_{\pi}(\sigma) \]  \[ (4.15) \]
  \[ \langle C_\sigma(T\eta)(\sigma, i), C_\sigma(T\eta)(\sigma, j) \rangle = 0, \text{ for } 1 \leq i < j \leq m_{\pi}(\sigma) \]  \[ (4.16) \]
Note that conditions (4.15) and (4.16) implicitly contain that \((T\eta)(\sigma, i) \in \text{dom}(C_\sigma)\). Clearly, for the admissibility conditions to be explicit, \(T\), the \(C_\sigma\) and \(\frac{d\nu}{d\nu_G}\) need to be known explicitly. The next chapter is devoted to carrying this program out for a rather general setting.

In the simplest possible case, i.e., \(G\) unimodular, \(\pi\) multiplicity-free and \(\tilde{\nu} = \nu_G\), the admissibility condition reduces to

\[
\|\eta\| \equiv 1, \nu_G - \text{almost everywhere on } \Sigma. \quad (4.17)
\]

We close the section with a partial converse of the results in the last remark, at least for unimodular groups: Any admissibility condition which has a similar structure as the admissibility criterion (4.15) and (4.16) necessarily has to coincide with it. Also, the admissibility condition characterizes the canonical Plancherel measure. We expect this statement to hold for arbitrary groups, but have preferred only to deal with the unimodular case and to avoid the problems that arise from the Duflo-Moore operators. For certain representations however, an analogous result is derived in Theorem 5.23.

In any case the next theorem provides further evidence for the central thesis of this book, namely that computing admissible vectors is in a sense equivalent to computing Plancherel measure. The following subsection will show that this observation actually extends to the type I condition.

**Theorem 4.31.** Let \(G\) be unimodular. Suppose that

\[
\pi = \int_X ^{(\oplus)} m(x)\sigma_x \, d\nu(x)
\]

is a direct integral representation, where \(X\) is a standard Borel space with \(\sigma\)-finite measure \(\nu\), \(m : X \to \mathbb{N} \cup \{\infty\}\) is a Borel map and \((\sigma_x)_{x \in X}\) is a measurable field of representations of \(G\). Assume that the admissibility criterion

\[
(\eta^i_x)_{x \in X, i = 1, \ldots, m(x)} \text{ is admissible} \iff (\eta^i_x)_{i = 1, \ldots, m(x)} \text{ is an ONS for } \nu\text{-a.e. } x \in X. \quad (4.19)
\]

holds and that there exist vectors \(\eta\) fulfilling it.

(a) There exists a \(\nu\)-conull subset \(X' \subset X\) such that \(x \mapsto \sigma_x\) is a Borel embedding \(X' \hookrightarrow \hat{G}\).

(b) In this identification, \(\nu\) is the restriction of \(\nu_G\) to a suitable subset of \(\hat{G}\).

**Proof.** Let \(\pi_0 = \int_X ^{(\oplus)} \sigma_x d\nu(x)\), and \(\mathcal{H}_{\pi_0}\) the associated representation space. We first prove that (4.19) entails
\[ \|V_\eta \phi\|_2^2 = \int_X \|\eta_x\|^2 \|\phi_x\|^2 d\nu(x) , \]  

(4.20)

for all vector fields \( \eta = (\eta_x)_{x \in X}, \phi = (\phi_x)_{x \in X} \in H_{\pi_0} \), in the extended sense that \( V_\eta \phi \not\in L^2(G) \) whenever the right hand side is infinite. For this purpose define the auxiliary vector field \( \tilde{\eta} \in H_\pi \), by letting

\[ \tilde{\eta}_{x,i} = \|\eta_x\|^{-1} \eta_x , \]

and choosing \( \tilde{\eta}_{x,i}, i = 2, \ldots, m(x) \) orthonormal to each other and to \( \eta_x \), whenever \( \eta_x \neq 0 \). For those \( x \) for which \( \eta_x = 0 \), we pick an arbitrary orthonormal system \( \tilde{\eta}_{x,i}, i = 1, \ldots, m(x) \) (arbitrary within the measurability requirement, that is). Note that the assumption that there exists a square-integrable vector field fulfilling the admissibility condition implies that \( m(x) \leq \dim(H_x) \), as well as

\[ \int_X m(x) \, d\nu(x) < \infty . \]

Hence \( \tilde{\eta} \) can be constructed and is square-integrable as well. Moreover it is admissible by assumption. Next define a vector field \( \tilde{\phi} \) by

\[ \tilde{\phi}_{x,i} = \begin{cases} \|\eta_x\| \phi_x & i = 1 \\ 0 & i = 2, \ldots, m(x) \end{cases} , \]

and assume for the moment that \( \tilde{\phi} \) is square-integrable, i.e., in \( H_\pi \). Then

\[ V_{\tilde{\eta}} \tilde{\phi}(y) = \int_X \sum_{i=1}^{m(x)} \langle \tilde{\phi}_{x,i}, \sigma_x(y) \tilde{\eta}_{x,i} \rangle d\nu(x) \]

\[ = \int_X \langle \phi_x, \sigma_x(y) \eta_x \rangle d\nu(x) \]

\[ = V_\eta \phi(y) . \]

Hence admissibility of \( \tilde{\eta} \) entails

\[ \|V_\eta \phi\|_2^2 = \|\tilde{\phi}\|^2 = \int_X \|\eta_x\|^2 \|\phi_x\|^2 d\nu(x) . \]

This proves (4.20) for the case that the right-hand side is finite; the general case is easily obtained by plugging in restrictions of \( \phi \) to suitable Borel subsets \( B \subset X \).

Now part (a) follows from Theorem 4.32 below. For part (b) we note that by part (a) (4.18) is a decomposition into irreducibles, hence Theorem 3.25 yields that \( \nu_G \) and \( \nu \) are equivalent.

Moreover, we can conclude in the same way that (4.20) holds also with \( \nu_G \) replacing \( \nu \), by the admissibility criterion (4.17). Plugging in \( \eta \) with \( \|\eta_x\| = 1 \) almost everywhere yields

\[ \int_X \|\varphi_x\|^2 d\nu_G(x) = \int_X \|\varphi_x\|^2 d\nu(x) \]

for all vector fields \( \varphi \). Hence \( \nu = \nu_G \).
4.4 Admissibility Criteria and the Type I Condition

In this section we comment on the relation of the type I property to the existence of admissibility conditions. A major motivation for tackling non-type I groups is provided by discrete $G$: Here $\lambda_G$ is type I iff $G$ itself is \cite{70}, and the latter is the case only if $G$ is a finite extension of an abelian normal subgroup \cite{111}, i.e., very rarely.

The following theorem shows that for any direct integral representation $\pi$ with associated admissibility conditions as in Remark 4.30, $\pi$ is necessarily type I. Thus admissibility criteria outside the type I setting will have to be of a different nature.

**Theorem 4.32.** Let $G$ be unimodular, not necessarily of type I. Suppose that $\pi = \int_X m(x)\sigma_x \, d\nu(x)$ is a direct integral representation, where $X$ is a standard Borel space with $\sigma$-finite measure $\nu$, $m : X \to \mathbb{N} \cup \{\infty\}$ is a Borel map and $(\sigma_x)_{x \in X}$ is a measurable field of representations of $G$. Assume that the admissibility criterion
\[ (\eta^i_x)_{x \in X, i=1,\ldots,m(x)} \text{ is admissible} \iff (\eta^i_x)_{i=1,\ldots,m(x)} \text{ is an ONS for } \nu\text{-a.e. } x \in X. \quad (4.21) \]
holds and that there exist vectors $\eta$ fulfilling it. Then $\pi$ is type I.

**Proof.** Just as in the proof of Theorem 4.31, let $\pi_0 = \int_X \sigma_x \, d\nu(x)$, and $\mathcal{H}_{\pi_0}$ the associated representation space. Recall from the proof of 4.31 that
\[ \|V_\eta \phi\|_2^2 = \int_X \|\eta_x\|^2 \|\phi_x\|^2 \, d\nu(x). \quad (4.22) \]
Observe that the proof of that equation did not rely on the type I property of $\pi$.

We are going to show that $\pi_0$ is multiplicity-free. Since $\pi \approx \pi_0$, the type I property of $\pi$ then follows immediately. Let $\mathcal{K} \subset \mathcal{H}_{\pi_0}$ be an invariant subspace. $\pi$ is cyclic by assumption, hence there exists a cyclic vector $\psi$ for the subspace $\mathcal{K}$ as well. Then (4.22) allows to compute the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}_{\pi_0}$ as
\[ \mathcal{K}^\perp = \{ \phi : V_\psi \phi = 0 \} = \{ \phi : \|\phi_x\|\|\psi_x\| = 0, \nu - \text{almost everywhere} \} . \]
But this entails that the projection onto $\mathcal{K}$ is given by pointwise multiplication with the characteristic function of $\{ x \in X : \psi_x \neq 0 \}$. It follows that all invariant projections commute, and thus the commuting algebra of $\pi_0$ is commutative.

**Remark 4.33.** If $\pi$ is as in the theorem, then $\pi$ is a subrepresentation of $\lambda^I_G$, the type I part of $\lambda_G$, and $\nu$ is absolutely continuous with respect to the measure $\nu_G$ on $\hat{G}$ underlying the decomposition of $\lambda^I_G$ into irreducibles.
We are not aware of a systematic treatment of admissibility conditions outside the type I setting. There exists a substitute for the Plancherel theorem, see e.g. [35, 18.7.7], which uses the theory of traces and their decomposition. Some use of this result can be made to formulate admissibility criteria in the general setting, see [55].

4.5 Wigner Functions Associated to Nilpotent Lie Groups**

In this section we sketch the construction of Wigner functions associated to certain representations of nilpotent Lie groups. A full understanding of the results requires knowledge of Kirillov’s theory of coadjoint orbits and their use in constructing the unitary dual. We refrain from giving an introduction to this theory here, and refer the interested reader to [30, 72] for details. While the results only hold in a specific context, they serve as an example how representation theory can provide a unified view of phenomena connected to continuous wavelet transforms. Also, several ideas that have occurred in the discussion so far make their appearance in this section: The extension of orthogonality relations to Hilbert-Schmidt-operators, as used in the proof of 2.33, or the problem of decomposing operators over a direct integral space. The latter problem will be seen to connect nicely to Kirillov’s orbit theory.

The Wigner transform is intended as a symbol calculus associating to an operator on a certain Hilbert space a function on a phase space, i.e., to an observable in the quantum mechanical sense an observable in the classical sense. The original Wigner transform is closely related to the Heisenberg group (see the next subsection). The authors of [24] demonstrated how one could replace the Heisenberg group by the affine group and thus arrive at a different symbol calculus. As a matter of fact, there are various constructions based on different choices of groups around, see for instance [11, 12, 96, 24, 2]. In the following we will be concerned only with [2] and a variation of the construction, presented in [4]. The appeal of the approach presented here derives from the way it highlights the role of the Plancherel transform in the construction. The relations between this construction and the various other definitions of symbol calculi to be found in the literature are not entirely clear to us, and we do not have a particular claim to originality.

The authors of [2] singled out two main ingredients of the construction: A discrete series representation of the underlying group, with the associated orthogonality relations, and a Euclidean Fourier transform on the group obtained from an identification with its Lie algebra via the exponential map. [4] then showed how this construction generalizes to cases where discrete series representations are not available. We present a discussion which is suited to simply connected, connected nilpotent Lie groups; extensions to exponential Lie groups are possible.
The Original Wigner Transform

The starting point of the construction is the Heisenberg group $\mathbb{H}$. As a set, $G = \mathbb{R}^3$, with the group law

$$(p, q, t)(p', q', t') = (p + p', q + q', t + t' + (q'p - qp')/2).$$

The comparison with Example 2.27 shows that the reduced Heisenberg group $H_r$ is the quotient group of $\mathbb{H}$ by the discrete central subgroup $\{0\} \times \{0\} \times \{2\pi \mathbb{Z}\}$. For the time being, this difference does not really matter, since in this subsection we are dealing with the windowed Fourier transform and ignore the action of the center $\{0\} \times \{0\} \times \mathbb{R}$. However, for the connection to Kirillov’s theory later on it is crucial to have a simply connected group.

Recall that

$$W_{fg}(p, q) = \int_{\mathbb{R}} g(x) f(x + p)e^{-2\pi i q(x + p/2)} dx.$$

Now the fact that $W_f$ is isometric iff $\|f\| = 1$ entails that

$$\|W_{fg}\|_{L^2(\mathbb{R}^2)} = \|f\|_2 \|W_{fg}\| = \|f\|_2 \|g\|_2,$$

for arbitrary $0 \neq f, g \in L^2(\mathbb{R})$. Then polarization provides the biorthogonality relation

$$\langle W_{fg_1}, W_{fg_2} \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle; \quad (4.23)$$

similar arguments were used in connection with the orthogonality relations (2.15) for general discrete series representations. Since the right-hand side is nothing but the Hilbert-Schmidt scalar product of the two rank-one operators $g_1 \otimes f_1$ and $g_2 \otimes f_2$, we find that the mapping

$$W : (f, g) \mapsto W_{fg}$$

linearly extends to a unique isometry $W : B_2(L^2(\mathbb{R})) \to L^2(\mathbb{R}^2)$. As a matter of fact the map is onto, hence unitary. Now the Wigner transform, which we denote by $\mathcal{W}$, is obtained by taking the usual scalar-valued Plancherel transform after $W$, i.e., formally

$$\mathcal{W}(g \otimes f)(p^*, q^*) = \int_{\mathbb{R}} \int_{\mathbb{R}} W_{fg}(p, q)e^{-2\pi i (p^*p + q^*q)} dp dq = \int_{\mathbb{R}} e^{-2\pi ip^*} g \left(-q^* - \frac{p}{2}\right) f \left(-q^* + \frac{p}{2}\right) dp. \quad (4.24)$$

Here the second equality is Fourier inversion.

Let us collect the main properties of the operator $\mathcal{W} : B_2(L^2(\mathbb{R})) \to L^2(\mathbb{R}^2)$:
• **Unitarity**, usually expressed in the **overlap condition** \([2]\), or **Moyal’s identity**

\[
\langle \mathcal{W}(g_1 \otimes f), \mathcal{W}(g_2 \otimes f_2) \rangle = \langle g_1, g_2 \rangle \langle f_2, f_1 \rangle.
\]

• **Covariance:** If \(A \in B_2(L^2(\mathbb{R}))\), then

\[
\mathcal{W}(\pi(p, q, t)A\pi(p, q, t)^*)(p', q') = \mathcal{W}(A)(p' - p, q' - q).
\]

This is verified by direct calculation for rank-one operators and extends by density to arbitrary Hilbert-Schmidt operators.

• **Reality:** If \(A \in B_2(L^2(\mathbb{R}))\), then

\[
\mathcal{W}(A^*) = \overline{\mathcal{W}(A)}.
\]

Again, this is easily seen for rank-one operators.

Note that our calculations in this subsection are somewhat ad hoc. In particular the group-theoretic significance is not clear, since we considered restrictions to the subset \(\mathbb{R}^2 \times \{1\} \subset G\), which is not a subgroup. The construction of the Fourier-Wigner operator from (restrictions of) wavelet coefficients closely resembles Plancherel inversion. This observation will be the basis of the general approach for the construction of a transform with the above three properties, which works for arbitrary simply connected nilpotent groups.

### Wigner Functions Associated to Nilpotent Lie Groups

Let \(N\) be a simply connected, connected nilpotent Lie group, with Lie algebra \(\mathfrak{n}\). The basic facts concerning these groups, as used in the following, are contained in \([30]\). \(N\) is an **exponential** Lie group, i.e., the exponential mapping \(\exp : \mathfrak{n} \to N\) is bijective. Moreover, the image measure under \(\exp\) of Lebesgue measure on \(\mathfrak{n}\) turns out to be left- and rightinvariant on \(N\) \([30, \text{Proposition 1.2.9}]\). Let \(L^2(\mathfrak{n})\) denote the \(L^2\)-space with respect to Lebesgue measure, then we obtain a unitary map \(\text{EXP}^* : L^2(N) \ni f \mapsto f \circ \exp \in L^2(\mathfrak{n})\). Finally, we let \(\mathcal{P}_n : L^2(\mathfrak{n}) \to L^2(\mathfrak{n}^*)\) denote the usual Euclidean Plancherel transform, defined on \(L^2(\mathfrak{n}) \cap L^1(\mathfrak{n})\) by

\[
\mathcal{P}_n(f)(X^*) = \int_{\mathfrak{n}} f(X)e^{-i\langle X, X^* \rangle}dX,
\]

where \(dX\) is Lebesgue measure and \(\langle X, X^* \rangle\) denotes the duality between \(\mathfrak{n}\) and \(\mathfrak{n}^*\). For a suitable normalization of Lebesgue measure on \(\mathfrak{n}^*\) the map is unitary. Now we define the **global Wigner transform** \(\mathcal{W} : B_2^{\oplus} \to L^2(\mathfrak{n}^*)\) as

\[
\mathcal{W} = \mathcal{P}_n \circ \text{EXP}^* \circ \mathcal{P}^{-1}
\]

The construction entails the following theorem. We let \(\text{Ad}\) and \(\text{Ad}^*\) denote the adjoint resp. coadjoint actions of \(N\) on \(\mathfrak{n}\) resp. \(\mathfrak{n}^*\). \(\text{Ad}\) is obtained by differentiating the conjugation action of \(N\) on itself, which gives rise to a linear action of \(N\) on \(\mathfrak{n}\). \(\text{Ad}^*\) is obtained by transposing this action.
Theorem 4.34. (a) \( \mathfrak{W} \) is unitary.
(b) \( \mathfrak{W} \) is covariant: For all \((A_\sigma)_{\sigma \in \hat{N}} \in B_2^\oplus\), for all \(x \in N\),
\[
\mathfrak{W}((\sigma(x)A_\sigma \sigma(x)^*)_{\sigma \in \hat{N}})(X^*) = \mathfrak{W}(\text{Ad}^*(x)X^*) \text{ almost everywhere.}
\]
(c) \( \mathfrak{W} \) is real: For all \((A_\sigma)_{\sigma \in \hat{N}} \in B_2^\oplus\),
\[
\mathfrak{W}((A_\sigma^*)_{\sigma \in \hat{N}})(X^*) = \overline{\mathfrak{W}((A_\sigma)_{\sigma \in \hat{N}})(X^*)}.
\]

Proof. Part (a) is obvious from the definition. For part (b) we first use the intertwining property of the Plancherel transform to see that, for \(a = P^{-1}((A_\sigma)_{\sigma \in \hat{N}})\),
\[
P^{-1}((\sigma(x)A_\sigma \sigma(x)^*)_{\sigma \in \hat{N}})(y) = a(x^{-1}yx).
\]
Moreover, we have the fundamental relation (see [30])
\[
\exp((\text{Ad})(x)Y) = x \exp(Y) x^{-1},
\]
as well as
\[
\mathcal{P}_n(f \circ \text{Ad}(x))(X^*) = \int_n f(\text{Ad}(x)Y)e^{i(Y,X^*)}dX
\]
\[
= \int_n f(Y)e^{i(\text{Ad}(x^{-1})Y,X^*)}dX
\]
\[
= \int_n f(Y)e^{i(Y,\text{Ad}^*(x)X^*)}dX
\]
\[
= \mathcal{P}_n(f)(\text{Ad}^*(x)X^*).
\]
Combining these covariances gives (b). For part (c) observe that by Lemma 4.14 (iv)
\[
\mathcal{P}_N(a^*) = (A_\sigma^*)_{\sigma \in \hat{N}},
\]
and we recall that \(a^*(x) = \overline{a(x^{-1})}\). Since \(\exp(-X) = \exp(X)^{-1}\), \(\text{EXP}^*\) intertwines the involution on \(L^2(N)\) with the analogous involution on \(\mathfrak{n}\), viewed as a vector group. But this implies (c).

While this result was pleasantly simple to prove, it is still not clear how to recover the original Wigner transform from it; after all the latter maps single Hilbert-Schmidt operators to functions in two variables, not fields of operators to functions in three variables. As will be seen below, the missing link is provided by the role of the coadjoint orbits in the computation of \(\hat{N}\) and \(\mu_N\). This brings up Kirillov’s orbit method.

The central result of harmonic analysis on nilpotent Lie group is the existence of the Kirillov correspondence, which is a bijection
\[
\kappa : \mathfrak{n}^*/\text{Ad}^*(N) \to \hat{N}.
\]
In fact it is a Borel isomorphism, if we endow the right hand side with the Mackey Borel structure and the left hand side with the quotient structure. We denote the inverse mapping by $\kappa^{-1}(\sigma) = O^*_\sigma$. The orbit space is countably separated, i.e., $\hat{N}$ is type I.

Besides providing a scheme to compute $\hat{N}$, or at least a parametrization of it, the coadjoint orbits also give access to the Plancherel measure: Since the orbit space is countably separated, there exists a measure decomposition

$$dX^* = d\mu_O(X^*)d\nu(O)$$

of Lebesgue measure on $n^*$ (see Proposition 3.28), and it can be chosen in such a way that the image measure of $\nu$ under $\kappa$ is precisely $\nu_N$ [72].

By Proposition 3.29 the measure decomposition gives rise to a direct integral decomposition of $L^2(n^*)$, namely

$$L^2(n^*,dX) \simeq \int_{n^*/\text{Ad}^*(G)}^{\oplus} L^2(O,d\mu_O)\ d\nu(O).$$

Now $\mathfrak{M}$ can be read as an operator between two direct integral spaces, (essentially) based on the same measure space, and the following definition is natural:

**Definition 4.35.** The Wigner transform $\mathfrak{M}$ decomposes if there exists a conull, $\text{Ad}^*(N)$-invariant Borel set $C \subset n^*$, and a field of operators

$$\mathfrak{M}_\sigma : B_2(\mathcal{H}_\sigma) \to L^2(O^*_\sigma,\mu_{O^*_\sigma})$$

for all $\sigma \in \hat{N}$ with $O^*_\sigma \subset C$, such that the following relation on $B_2^{\oplus}$ holds:

$$\mathfrak{M}((A_\sigma)_{\sigma \in \hat{N}}) = (\mathfrak{M}_\sigma(A_\sigma))_{\sigma \in \hat{N}}. \quad (4.25)$$

If the operators $\mathfrak{M}_\sigma$ exist, they are called **local Wigner transforms**.

Note that by definition the set of $\sigma \in \hat{G}$ for which $\mathfrak{M}_\sigma$ is not defined has measure zero, hence (4.25) is meaningful for $B_2^{\oplus}$.

**Remark 4.36.** (1) In the next subsection we will see that the concrete Wigner transform given above is a local Wigner operator in the sense of the definition. 

(2) The motivation for local Wigner operators is to obtain a correspondence between single operators and functions, instead of families of operators and (families of) functions. The desire to have Wigner functions supported by single coadjoint orbits is motivated by the applications of Wigner functions in mathematical physics. The Wigner transform is intended as a quantization procedure, assigning each operator on a given Hilbert space its symbol, i.e., a function on phase space. Now the symplectic structure on coadjoint orbits provides a natural interpretation of these orbits as phase spaces of physical systems, whereas $n^*$, as a disjoint union of such phase spaces, does not readily
4.5 Wigner Functions Associated to Nilpotent Lie Groups

lend itself to such an interpretation. For more details on quantization and coadjoint orbits, we refer the reader to [62, 72].

(3) The construction can be extended to exponential groups. The extra cost consists mainly in having to deal with densities when passing from $N$ to $\mathfrak{n}$, i.e., Haar measure of $N$ need not coincide with Lebesgue measure. Also the group can be nonunimodular. Confer [2, 3] for examples.

(4) The question whether $\mathfrak{W}$ decomposes has the following representation-theoretic background: The proof of the covariance property in Theorem 4.34 is based on the fact that $\mathcal{P}_n \circ \text{EXP}^*$ decomposes the conjugation representation

$$x \mapsto \lambda_G(x) \varrho_G(x)$$

with the unitary action on $L^2(\mathfrak{n}^*)$ induced by $\text{Ad}^*$, whereas $\mathcal{P}$ intertwines it with the direct integral representation

$$\int_{\mathfrak{n}^*/\text{Ad}^*(N)} \pi_O d\nu(O) , \tag{4.26}$$

where $\pi_O$ is a representation acting on $\mathcal{B}_2(\mathcal{H}_\kappa(O))$ via

$$\pi_O(x) A = \kappa(O)(x) \otimes \kappa(O)(x) .$$

Since both $L^2(\mathfrak{n}^*)$ and $\int_{\mathfrak{n}^*/\text{Ad}^*(N)} \mathcal{B}_2(\mathcal{H}_\kappa(O)) d\nu(O)$ can be viewed as direct integral spaces based on the same measure space, the decomposition statement is natural. It is nontrivial, however: The point is that the decomposition (4.26) is not easily related to the decomposition of the conjugation representation into irreducibles. In particular, $\pi_O$ may contain representations which correspond to coadjoint orbits other than $O$. Hence additional assumptions are necessary, as witnessed by the next theorem showing that the local Wigner operators exist only in very restrictive settings.

**Theorem 4.37.** Let $N$ be a simply connected, connected nilpotent Lie group. The Wigner transform on $N$ decomposes iff there exists a conull $\text{Ad}^*(N)$-invariant Borel subset $C \subset \mathfrak{n}^*$ such that every coadjoint orbit in $C$ is an affine subspace.

**Proof.** We first prove that $\mathfrak{W}$ decomposes iff there exists a conull $\text{Ad}^*(N)$-invariant Borel subset $C \subset \mathfrak{n}^*$ such that for all $f \in L^1(N) \cap L^2(N)$, and all $\sigma \in \hat{N}$ with $O_{\sigma}^* \subset C$,

$$\|\sigma(f)\|_2^2 = \int_{O_{\sigma}^*} |\mathcal{P}_n(f)(\omega)|^2 \, d\mu_{O_{\sigma}^*}(\omega) . \tag{4.27}$$

Indeed, supposing that $\mathfrak{W}$ decomposes, the right hand side is precisely $\|\mathfrak{W}_\sigma(\sigma(f))\|_2^2$, and unitarity of $\mathfrak{W}$ entails $\nu_N$-almost everywhere that $\mathfrak{W}_\sigma$ is unitary. This proves the “only-if” part.
For the “if-part” we construct $W^{-1}_\sigma$ as follows: Fix a coadjoint orbit $O^*_\sigma$, for $\sigma \in C$. By [30, 3.1.4], $O^*_\sigma$ is a closed submanifold of $n^*$. Pick a function $g \in C_c^\infty(n^*)$, then $P_n^{-1}(g) \in L^1(N) \cap L^2(N)$. Moreover, the restriction of $g$ to $O^*_\sigma$ is in $C_c^\infty(O^*_\sigma)$. We define the operator $G_\sigma \in B_2(H_\sigma)$

$$G_\sigma = \sigma(P_n^{-1}(g))$$

We claim that $G_\sigma$ only depends on the restriction $g|_{O^*_\sigma}$: Indeed, if $h \in C_c^\infty(n^*)$ fulfills $h|_{O^*_\sigma} = g|_{O^*_\sigma}$, then (4.27) yields

$$\|\sigma(P_n^{-1}(f - h))\|_2^2 = 0.$$ Hence the operator $M_\sigma : g|_{O^*_\sigma} \mapsto G_\sigma$ is well-defined. Moreover, the assumption yields $\|g|_{O^*_\sigma}\|_2^2 = \|G_\sigma\|_2^2$. Since $O^*_\sigma$ is a closed submanifold, the set of restrictions of $C_c^\infty$-functions is precisely $C_c^\infty(O^*_\sigma)$, which is dense in $L^2(O^*_\sigma)$. Hence $M_\sigma$ has a unique isometric extension $L^2(O^*_\sigma) \to B_2(H_\sigma)$. By construction, $M^{-1}$ thus coincides with the direct integral of the field $(M_\sigma)_{\sigma \in C}$, at least on $C_c^\infty(n^*)$. But this space is dense, hence we conclude that the $M_\sigma$ are unitary, and $M$ decomposes into the $M_\sigma = M^{-1}_\sigma$.

Hence the equivalence is established, and we can finish by appealing to [85, Theorem 1] which states that (4.27) holds iff $O^*_\sigma$ is an affine subspace.

**Remark 4.38.** There appears to be a way of defining local Wigner transforms outside the flat orbit case. It consists in replacing the operator $P_n$ by a different integral operator $P_{n}^{\text{ad}}$, the so-called **adapted Fourier transform** [10, 12, 13].

The transform is designed to ensure

$$\|\sigma(f)\|_2^2 = \int_{O^*_\sigma} |P_{n}^{\text{ad}}(f)(\omega)|^2 \, d\mu_{O^*_\sigma}(\omega),$$

and as in the proof of the theorem, this entails that the adapted Wigner transform

$$M^{\text{ad}} = P_{n}^{\text{ad}} \circ \text{EXP}^* \circ P^{-1}$$

decomposes. However, we have not been able to check whether the adapted transform has the covariance property.

**The Original Wigner Function Revisited**

Let us now compute the global Wigner transform for the Heisenberg group, and show how to recover the original Wigner transform from it. The formal calculations below can be made rigorous, either by restricting to suitable functions for which the integrals exist pointwise, or by weak arguments, i.e., taking scalar products with $L^2$-functions and using Plancherel’s theorem instead of Fourier inversion. We have refrained from doing either.

The Heisenberg Lie algebra of the group is $\mathfrak{h} = \mathbb{R}^3$, with Lie bracket
The associated Lie group $\mathbb{H}$ can be thought of as $\mathfrak{h}$, endowed with the Campbell-Baker-Hausdorff formula, which yields the group structure
\[(p, q, t)(p', q', t') = (p + p', q + q', t + t' + (pq' - p'q)/2) .\]

Note that taking for $\mathbb{H}$ the Lie algebra $\mathfrak{h}$ with Campbell-Baker Hausdorff formula amounts to taking the identity operator as exponential map. The adjoint representation of $H$ on $\mathfrak{h}$ is computed as
\[\text{Ad}(p, q, t)(p', q', t') = (p', q', t' + pq' - p'q) ,\]
and if $(p^*, q^*, t^*)$ are the coordinates with respect to the dual basis,
\[\text{Ad}^*(p, q, t)(p^*, q^*, t^*) = (p^* + t^*p, q^* + t^*q, t^*) .\]

It follows that the set of coadjoint orbits consists of the singletons $\{(p^*, q^*, 0)\}$, with $(p^*, q^*) \in \mathbb{R}^2$, and the hyperplanes $\mathcal{O}^*_{t^*} = \mathbb{R}^2 \times \{t^*\}$. The representation corresponding to $\mathcal{O}^*_{t^*}$ under the Kirillov map is the Schrödinger representation $\varrho_{-t^*}$ (observe the sign change). Here $g_{t^*}$ acts on $L^2(\mathbb{R})$ via
\[[g_{t^*}(p, q, t)f](x) = e^{2\pi i (q x + t^* t - t^* p q/2)} f(x + t^* p).\]

Now the Plancherel measure is seen to be supported by the Schrödinger representations, where it is given by $dv_{\mathbb{H}}(\sigma_{t^*}) = |t^*| dt^*$ [45, Section 7.6], with $dt^*$ denoting Lebesgue measure on the real line.

We have now collected all necessary ingredients for the computation of $\mathcal{M}$. Note that we know already by Theorem 4.37 that $\mathcal{M}$ decomposes, so we will also be interested in the components. Pick a measurable field of rank-one operators $R = (g_{t^*} \otimes f_{t^*})_{t^* \in \mathbb{R}} \in \mathcal{B}^\oplus_2 \cap \mathcal{B}^\oplus_1$. Then the Plancherel inversion formula yields for $r = P^{-1}(R)$ that
\[r(p, q, t) = \int_{\mathbb{R}} \langle g_{t^*}, \varrho_{-t^*}(p, q, t)f_{t^*} \rangle |t^*| dt^*\]
\[= \int_{\mathbb{R}} \int_{\mathbb{R}} g_{t^*}(x) f_{t^*}(x - t^* p) e^{-2\pi i (qx - t^* t - t^* p q/2)} |t^*| dt^*.\]

Plugging this into the Euclidean Plancherel transform yields formally
\[\mathcal{M}(R)(p^*, q^*, s^*) = \int_{\mathbb{R}^5} g_{t^*}(x) f_{t^*}(x - t^* p) e^{-2\pi i (qx - t^* t - t^* p q/2 + q^* q + p^* p + s^* t)} |t^*| dt^* dxdpqdt\]

Now the integration against $dxdq$ can be simplified by Fourier inversion to the substitution $x \mapsto -q^* + t^* p/2$, thus the integral becomes
\[
\int_{\mathbb{R}^3} g_t^*(-q^* + t^* p/2)f_t^*(-q^* - t^* p/2)e^{-i(t^* t + p^* p + s^* t)}|t^*| \, dt^* \, dp \, dt
\]

\[
= \int_{\mathbb{R}} g_s^*(-q^* + s^* p/2)f_s^*(-q^* - s^* p/2)e^{-i p^* p} |s^*| \, dp.
\]

Here another Fourier inversion allowed to discard the integral \(dt^* dp\), resulting in the substitution \(t^* \mapsto s^*\). Now a closer look at the remaining term reveals that indeed the values of \(\mathcal{M}(R)\) on the coadjoint orbit \(\mathbb{R}^2 \times \{s^*\}\) only depend on \(g_s^* \otimes f_s^*\), and that the local Wigner operators are given by

\[
[\mathcal{M}_s^* (g \otimes f)](p^*, q^*, s^*) = |s^*| \int_{\mathbb{R}} g \left(-q^* + \frac{s^* p}{2}\right) f \left(-q^* - \frac{s^* p}{2}\right) e^{-i p^* p} \, dp.
\]

Now a comparison with (4.24) yields that the original Wigner transform discussed above is just the local Wigner transform \(\mathcal{M}_1\).